STRATIFICATION OF CLOUD LAYERS IN A STABLE ATMOSPHERE

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ABSTRACT

In this paper an attempt is made to explain the phenomenon of cloud stratification which is occasionally observed in the atmosphere. The leading thought is that these stratifications are caused by internal gravity waves. The atmosphere is assumed to consist of two layers of compressible air. The lower layer has a constant lapse rate, and the upper layer is isothermal. The quasi-static assumption is made. This assumption is found to be justifiable in the range of waves under consideration. It is found that waves whose phase velocities are of the order of 10m. sec.$^{-1}$ are capable of producing three strata whose elevations fall within the limits of observed altitudes of tropospheric clouds. The proposed mechanism is also found to be consistent with the observed heights of some stratospheric clouds.

1. INTRODUCTION

It is occasionally observed that clouds tend to appear in two or three discrete stratified layers separated by clear air. This is especially true when the atmosphere is stable. When the atmosphere is in unstable equilibrium these layers tend to merge together and appear as deep cloud formations.

This cloud stratification is quite common on the lee sides of mountains where it has been explained as a wave phenomenon. See, for example, Scorer [7]. However, this phenomenon is not limited to mountainous regions, although it is more frequent there. It is observed over flat areas where no mountains are in the neighborhood. It may even be observed over open oceans. A very striking view from an airplane flying at high altitude is just these cloud stratifications which may be seen here and there when the clouds are limited in horizontal extent so that one can see through them.

In spite of the fact that these stratifications must have been recognized for many decades, the present writer does not know of any theoretical mechanism that has been proposed to explain them and to describe the conditions under which they may be observed. The present communication may be considered as a first attempt in this direction.

The leading thought in the present article is that these stratifications may also be the result of some atmospheric oscillations. It is obvious that clouds tend to form where upward vertical motion exists. Because the air is necessarily devoid of vertical motion at the ground, a permanent nodal plane is imposed there. Other nodal planes parallel to the ground may form at higher levels depending upon the mode of oscillation executed by the atmosphere. The atmosphere may therefore be visualized as being divided into regions of vertical activity separated by these nodal planes. Moreover, the sense of vertical motion must alternate from one region to the other, being upward in one and downward in the next. The maximum vertical activity occurs at the antinodes. Clouds are therefore expected to form in every other region provided that the humidity and other meteorological elements are adequate for their formation. Clear regions may be expected to be left where downward motion exists.

The purpose of the present paper is to study these oscillations and describe the conditions governing the separation of the consecutive antinodes.

2. MATHEMATICAL ANALYSIS

It will be assumed that the atmosphere consists of two compressible layers, a lower layer extending from the ground to the tropopause, and an upper layer resting on top of the first and extending to infinity. The lower layer may be characterized by a constant lapse rate, and the upper layer may be isothermal. Both layers may be at rest when they are undisturbed. Friction and the earth's spherical shape may be neglected. It is required to study the free gravitational oscillations of such an atmosphere and, in particular, to describe the vertical motions associated with transverse waves traveling in the horizontal direction.

The basic equations describing motion in this model, and pertaining to a fixed system of coordinates, have been derived by Lamb [5] and discussed by Taylor [9] in connection with the Krakatoa eruption of 1883. Fundamentally the same problem has been treated by Haurwitz [1], [2], who obtained a frequency equation for these waves under some simplifying assumptions. These equations have been generalized to include the earth's rotation and its spherical shape by Pekeris and others. See Wilkes [11]. More recently Martyn [6] has discussed the same problem in relation to waves in the ionosphere and troposphere. Sekera [8] has studied the effect of a wind shear on these waves when the earth's rotation and its spherical shape are neglected. Kuo [4] has discussed the stability conditions in a Couette flow. In the present treatment some numerical computations are made regarding vertical motions. The main point in view is to compute the vertical separations between the antinodes of the layers resulting from the oscillations. Lamb's notation and his general method of analysis are closely followed. One point of departure between the present analysis and that of Lamb is the introduction of the quasi-static hypothesis at the beginning of the analysis. This assumption is found to simplify the mathematical analysis considerably. At the same time, as will be shown in the Appendix, the introduction of this assumption is equivalent to neglecting some terms which are of higher order of magnitude than the retained terms. This assumption may therefore be justified.

Let the origin of a Cartesian system of coordinates be taken at the tropopause, the z-axis pointing upward and the x-axis horizontal and pointing in the direction of wave motion. The disturbed motion may be assumed to be independent of the y-direction. The ground may be assumed horizontal and at constant depth, h, below the tropopause. The undisturbed quantities may be independent of the horizontal directions.

The linearized equations describing the perturbed motion are the following. See Haurwitz [2].

The equation of motion is

$$\frac{\partial \mathbf{q}}{\partial t}+\mathbf{Q}\cdot\nabla\mathbf{q}+\mathbf{q}\cdot\nabla\mathbf{Q}+2\boldsymbol{\omega}\times\mathbf{q}=-\frac{1}{\rho_0}\nabla p+\frac{\rho}{\rho_0^2}\nabla P_0, \quad (1)$$

the equation of continuity is

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{Q})+\nabla\cdot(\rho_0\mathbf{q})=0, \quad (2)$$

and the equation of piezotropy is

$$c^2\left(\frac{\partial \rho}{\partial t}+\mathbf{Q}\cdot\nabla\rho+\mathbf{q}\cdot\nabla\rho_0\right)-\left(\frac{\partial p}{\partial t}+\mathbf{Q}\cdot\nabla p+\mathbf{q}\cdot\nabla P_0\right)=0 \quad (3)$$

where

$$c^2=\frac{\gamma P_0}{\rho_0}=\gamma RT \quad (4)$$

is the local speed of sound. In these equations $\mathbf{q}$, p, ρ are the perturbation velocity, pressure, and density respectively, $\mathbf{Q}$, P_0, ρ_0 are the corresponding undisturbed quantities, $\boldsymbol{\omega}$ is the angular velocity of the earth's rotation, γ is the ratio of the specific heats for dry air, R is the gas constant, and T is the temperature.

As already mentioned, in the present analysis the following assumptions are made: (a) $\mathbf{Q}=0$ and $\frac{\partial}{\partial y}=0$, (b) the change of state during the vibrations is adiabatic, and (c) the hydrostatic relation holds for both the undisturbed and the total disturbed motions.

With these assumptions the previous equations yield the following:

$$u_t-fv=-\frac{1}{\rho_0}p_x \quad (5)$$

$$v_t+fu=0 \quad (6)$$

$$0=-p_z-g\rho \quad (7)$$

$$\rho_t+w\rho_{0z}+\rho_0\chi=0 \quad (8)$$

$$c^2(\rho_t+w\rho_{0z})=p_t-wg\rho_0 \quad (9)$$

where

$$\chi=\nabla\cdot\mathbf{q}=u_x+w_z. \quad (10)$$

Here u, v, and w are the components of velocity; $f=2\omega \sin\theta$ is the Coriolis parameter. The subscripts t, x, and z indicate partial differentiations.

Upon the elimination of v, p, and ρ the following equations are obtained:

$$u_{tt}+f^2u=(c^2\chi-gw)_x, \quad (11)$$

and

$$(\gamma-1)g\chi+gw_z-c^2\chi_z=0. \quad (12)$$

Together with (10), equations (11) and (12) form a closed system from which u, w, and χ may be determined.

Because interest is centered around transverse waves that travel in the horizontal direction, the following form of solutions will be assumed:

$$\begin{vmatrix} u \\ w \\ \chi \end{vmatrix}=\begin{vmatrix} u(z) \\ w(z) \\ \chi(z) \end{vmatrix} e^{i(kx-\nu t)} \quad (13)$$

When these values are inserted in the last three equations the following equations are obtained:

$$u=-\frac{ik}{\sigma^2}(c^2\chi-gw) \quad (14)$$

$$\frac{g^2k^2}{\sigma^2}w=c^2\chi_z+\left(g\frac{k^2c^2}{\sigma^2}-\gamma g\right)\chi \quad (15)$$

$$c^2\chi_{zz}+\left(\frac{dc^2}{dz}-\gamma g\right)\chi_z+\frac{gk^2}{\sigma^2}\left[\frac{dc^2}{dz}+g(\gamma-1)\right]\chi=0 \quad (16)$$

where

$$\sigma^2=\nu^2-f^2. \quad (17)$$

The argument (z) has been dropped from the functions u, w, and χ since no confusion is expected.

The problem now reduces to solving (16) for χ, subject to the proper conditions, then substituting in (15) to obtain w.

In the isothermal stratosphere $T=T_0$, a constant, and

$c^2=\gamma RT_0=c_0^2$, a constant. Hence equation (16) takes the following form:

$$c_0^2\chi_{zz}-\gamma g\chi_z+\frac{g^2k^2}{\sigma^2}(\gamma-1)\chi=0. \tag{18}$$

For the troposphere, let the constant lapse rate be Γ. Then $T=T_0-\Gamma z$ and $c^2=\gamma R(T_0-\Gamma z)$. Let

$$\frac{g}{R\Gamma}=n+1 \tag{19}$$

and $\Gamma_a=\frac{g(\gamma-1)}{\gamma R}$, the adiabatic lapse rate.

Insertion of these quantities in (16) reduces it to the following form:

$$(T_0-\Gamma z)\chi_{zz}-\Gamma(n+2)\chi_z+\frac{gk^2}{\sigma^2}(\Gamma_a-\Gamma)\chi=0. \tag{20}$$

Upon making the following change of independent variable

$$\zeta=\frac{T_0}{\Gamma}-z \tag{21}$$

equation (20) takes the form:

$$\zeta\chi_{\zeta\zeta}+(n+2)\chi_\zeta+\frac{gk^2}{\sigma^2}\left(\frac{\Gamma_a}{\Gamma}-1\right)\chi=0. \tag{22}$$

It is clear that the natures of the solutions of equations (18) and (22) depend upon the sign of the quantity σ^2. Thus two distinct cases will be discussed. The first case is that in which $\nu>f$ so that $\nu^2>>f^2$ and $\sigma^2\approx\nu^2$. The second case is that in which $\nu<f$ so that $\nu^2<<f^2$ and $\sigma^2\approx -f^2$.

For waves whose phase velocities are of the order of 10^3 cm. sec.$^{-1}$, and in the middle latitudes where f is of the order of 10^{-4} sec.$^{-1}$, it can be seen that the first approximation holds for wavelengths of the order of 60 km. or less, whereas the second approximation holds for longer waves.

3. WAVES OF THE FIRST KIND IN AN ISOTHERMAL ATMOSPHERE

Upon replacing σ^2 by ν^2 and noting that $\nu/k=V$ is the phase velocity of the waves, equations (15) and (18) take the following forms respectively:

$$w=\frac{c_0^2V^2}{g^2}\chi_z+\frac{1}{g}(c_0^2-\gamma V^2)\chi \tag{23}$$

and

$$c_0^2\chi_{zz}-g\gamma\chi_z+\frac{g^2}{V^2}(\gamma-1)\chi=0 \tag{24}$$

The appropriate solution of equation (24) that satisfies the boundary condition of making the energy transport upward is (see Weekes and Wilkes [10]):

$$\chi=D_1e^{\frac{g}{c_0^2}\left(\frac{\gamma}{2}+ib\right)z} \tag{25}$$

where

$$b=\left[\left(\frac{c_0}{V}\right)^2(\gamma-1)-\left(\frac{\gamma}{2}\right)^2\right]^{1/2} \tag{26}$$

and D_1 is a constant of integration.

The condition to be satisfied in order to obtain periodic motion in the vertical is that b must be real. This condition is realized if the following inequality is satisfied:

$$\frac{V}{c_0}<\frac{2(\gamma-1)^{1/2}}{\gamma}. \tag{27}$$

The value of γ for dry air is 7/5. Insertion of this value in equation (27) gives the following limit for V/c_0:

$$\frac{V}{c_0}\leq\frac{2\sqrt{10}}{7}\approx 0.9.$$

This condition is always satisfied for waves whose velocities are in the neighborhood of the magnitudes normally observed for cloud movements.

Upon substituting from equation (25) in (23) the real part of w is found to be

$$w=\frac{D_1}{g}e^{\frac{\gamma g}{2c_0^2}z}\left[\left(c_0^2-\frac{1}{2}\gamma V^2\right)\cos\left(\frac{bg}{c_0^2}z\right)-V^2b\sin\left(\frac{bg}{c_0^2}z\right)\right]. \tag{28}$$

The vertical velocity therefore disappears at certain nodal planes whose heights z_s above the tropopause are given by the relation

$$\tan\left(\frac{bg}{c_0^2}z_s\right)=\frac{1}{b}\left[\left(\frac{c_0}{V}\right)^2-\frac{\gamma}{2}\right].$$

Hence

$$\frac{bg}{c_0^2}z_s=\theta_0+s\pi;\ s=0, 1, 2\ldots \tag{29}$$

where θ_0 is the angle corresponding to $s=0$. Thus it appears from equation (29) that the isothermal stratosphere may be divided into horizontal layers in which vertical velocities exist, and which are separated by nodal planes where vertical velocities vanish. The sense of vertical motion naturally alternates from one layer to the next. Because clouds are expected to form where the vertical motion is upward, the cloud strata that may correspond to these motions are separated at double the distance between two nodal planes. Hence the separation, l, between consecutive cloud layers is given by the following relation

$$l=2(z_{s+1}-z_s)=\frac{2\pi c_0^2}{gb}. \tag{30}$$

Figure 1 is a plot of the separation l against the nondimensional ratio V/c_0. The temperature of the isothermal stratosphere has been assumed to be 216° K. The speed of sound corresponding to this temperature is 2.91×10^4 cm. sec.$^{-1}$ It may be seen from this figure that the separation l is zero for wave velocity $V=0$, which is the case of no wave motion. l increases monotonously as V/c_0 increases, and becomes infinite when V/c_0 attains the limiting value of 0.9. Two cases may be of special interest. First, for a slowly moving wave whose velocity is of the order of 0.1 c_0, it is found that $l=8.6$ km. If the tropopause is at a height of 11 km., and if it coincides with a maximum vertical velocity, the usual case as will be shown later, then the first cloud stratum that may appear in the stratosphere will be at an elevation of 19.6 km. This is of the same order of magnitude as that of the elevations of nacreous clouds which are known to be slowly moving clouds. See [3], p. 385. Next consider a fast moving wave whose velocity is in the range ½ c_0 to ⅔ c_0. From figure 1 the values of l that correspond to this range are found to lie between 53 and 85 km. Hence the first cloud stratum in the stratosphere must appear at an elevation lying in the range 64 to 96 km. This again is of the same order of magnitude as that of the elevations of noctilucent clouds which are known to be fast moving clouds. See [3], p. 392.

4. WAVES OF THE FIRST KIND IN A TROPOSPHERE WITH CONSTANT LAPSE RATE

For the troposphere, where the lapse rate Γ has been assumed constant, equation (22) holds. This equation may be written in the form

$$\zeta\chi_{\zeta\zeta}+(n+2)\chi_\zeta+m\chi=0 \tag{22a}$$

where

$$m=\frac{g}{V^2}\left(\frac{\Gamma_a}{\Gamma}-1\right). \tag{31}$$

Upon making the following changes of variables

and

$$\left.\begin{aligned}\chi&=\zeta^{-\frac{n+1}{2}}\psi \quad &(a)\\ \eta^2&=4m\zeta \quad &(b)\end{aligned}\right\} \tag{32}$$

equation (22a) becomes

$$\psi''+\frac{1}{\eta}\psi'+\left[1-\frac{(n+1)^2}{\eta^2}\right]\psi=0 \tag{33}$$

where primes indicate differentiation with respect to η. Equation (33) is in the typical form of Bessel equation. The solution is

$$\psi=A_1J_{n+1}(\eta)+B_1Y_{n+1}(\eta) \tag{34}$$

where J and Y are the two kinds of Bessel functions, and A_1 and B_1 are the two constants of integration.

Upon inserting equation (34) in (32), then using equation (15), the following values are found for χ and w:

$$\chi=(4m)^{\frac{n+1}{2}}\eta^{-(n+1)}[A_1J_{n+1}(\eta)+B_1Y_{n+1}(\eta)] \tag{35}$$

$$w=\frac{\gamma V^2}{2(n+1)g}(4m)^{\frac{n+1}{2}}\eta^{-n}\left\{A_1\left[\frac{g}{2mV^2}\eta J_{n+1}(\eta)-J_n(\eta)\right]+B_1\left[\frac{g}{2mV^2}\eta Y_{n+1}(\eta)-Y_n(\eta)\right]\right\}. \tag{36}$$

The boundary condition at the ground is that the vertical velocity w must vanish. That is

$$w=0;\ \text{at } z=-h. \tag{37}$$

The value of the variable η at the ground may be obtained from equations (32) and (21), namely,

$$\eta_{-h}=\left[4m\left(\frac{T_0}{\Gamma}+h\right)\right]^{1/2},\ \text{at } z=-h. \tag{38}$$

Substituting from equations (37) and (38) in (36) yields the following relation between the constants B_1 and A_1:

$$\frac{B_1}{A_1}=C_1=-\frac{\left[\frac{g}{2mV^2}\eta_{-h}J_{n+1}(\eta_{-h})-J_n(\eta_{-h})\right]}{\left[\frac{g}{2mV^2}\eta_{-h}Y_{n+1}(\eta_{-h})-Y_n(\eta_{-h})\right]}. \tag{39}$$

Equation (36) now takes the form:

$$w=\frac{\gamma V^2(4m)^{\frac{n+1}{2}}}{2(n+1)g}\eta^{-n}A_1\left\{\left[\frac{g}{2mV^2}\eta J_{n+1}(\eta)-J_n(\eta)\right]+C_1\left[\frac{g}{2mV^2}\eta Y_{n+1}(\eta)-Y_n(\eta)\right]\right\}. \tag{40}$$

Equation (40) is identical with that obtained by Taylor [9] who derived it by making some approximations rather than by using the hydrostatic assumption.

Let the quantity ϕ be defined as

$$\phi\equiv-\left[\frac{\frac{4}{\sqrt{\pi}}(n+1)g\eta^{(n-\frac{1}{2})}}{\gamma V^2(4m)\,\frac{n+1}{2}}\right]w. \tag{41}$$

From this and equation (40) the following is obtained:

$$\phi=-\left[j_n(\eta)-\frac{g}{2mV^2}\eta j_{n+1}(\eta)\right]+C_1\left[y_n(\eta)-\frac{g}{2mV^2}\eta y_{n+1}(\eta)\right] \tag{42}$$

where j and y are the spherical Bessel functions of the two kinds. It may be seen from equation (41) that the function ϕ varies with w, such that the zeroes of ϕ are also the zeroes of w. A plot of ϕ against z is also an indication of the variation of w with z.

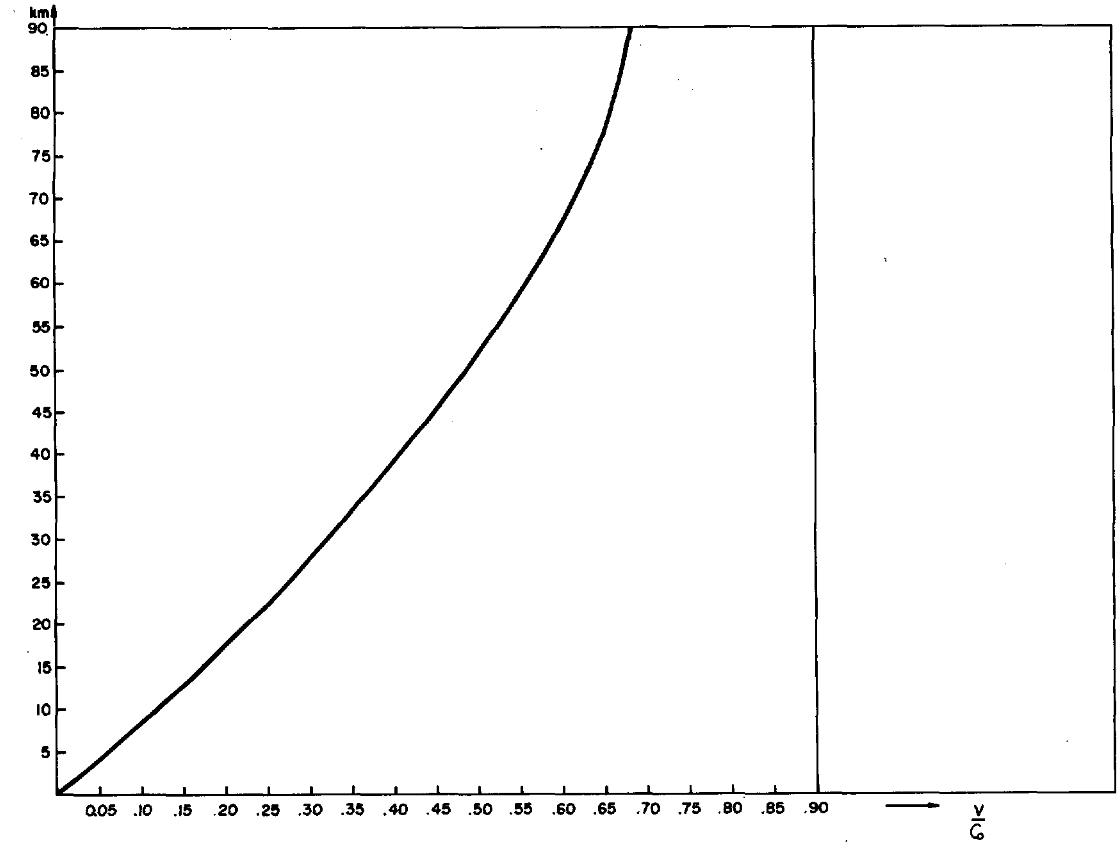

FIGURE 1.—A plot of the separation between cloud strata, l, against V/c_0 in an isothermal atmosphere. l is measured in kilometers and V/c_0 is a dimensionless number. The temperature of the isothermal atmosphere is assumed to be 216° K.

5. THE EIGENVALUES OF V

In order to plot the values of ϕ, as given by equation (42), against height it is first necessary to find the permissible values that V may take, namely the eigenvalues of this function. To do so the solutions for the stratosphere and the troposphere must be fitted together. It is clear that the boundary conditions, necessitated by the continuity of pressure and vertical velocity at the tropopause, are

$$\text{and}\qquad \left.\begin{aligned} w_1&=w_2 \quad \text{(a)}\\ p_1&=p_2 \quad \text{(b)}\end{aligned}\right\};\ \text{at } z=0 \tag{43}$$

where the subscripts 1 and 2 refer to the troposphere and stratosphere respectively. It can easily be shown that, when the temperature is continuous across the tropopause, the second condition is equivalent to

$$\chi_1=\chi_2;\quad \text{at } z=0. \tag{44}$$

Upon combining these conditions with equations (25), (28), (35), and (40), and making use of the relation (39), the following equation is obtained:

$$\begin{aligned}\left(c_0^2-\frac{1}{2}\gamma V^2\right)&\left\{J_{n+1}\left(\frac{2}{V\Gamma}\,(gT_0(\Gamma_a-\Gamma))^{1/2}\right)\right.\\ &\left.+C_1(V)Y_{n+1}\left(\frac{2}{V\Gamma}\,(gT_0(\Gamma_a-\Gamma))^{1/2}\right)\right\}\\ &-\frac{\gamma V}{(n+1)\Gamma}\,(gT_0(\Gamma_a-\Gamma))^{1/2}\left\{\left[\frac{1}{V}\left(\frac{gT_0}{\Gamma_a-\Gamma}\right)^{1/2}J_{n+1}\right.\right.\\ &\left.\times\left(\frac{2}{V\Gamma}\,(gT_0(\Gamma_a-\Gamma))^{1/2}\right)-J_n\left(\frac{2}{V\Gamma}\,(gT_0(\Gamma_a-\Gamma))^{1/2}\right)\right]\\ &+C_1(V)\left[\frac{1}{V}\left(\frac{gT_0}{\Gamma_a-\Gamma}\right)^{1/2}Y_{n+1}\left(\frac{2}{V\Gamma}\,(gT_0(\Gamma_a-\Gamma))^{1/2}\right)\right.\\ &\left.\left.-Y_n\left(\frac{2}{V\Gamma}\,(gT_0(\Gamma_a-\Gamma))^{1/2}\right)\right]\right\}=0\end{aligned} \tag{45}$$

where T_0 is the temperature at the tropopause and c_0 is the speed of sound corresponding to that temperature. It is to be noted that the quantity C_1, as given by (39) is a function of V.

Equation (45) is to be satisfied by the eigenvalues of V. The numerical procedure in finding the roots of this equation is to let the left-hand side equal to a function Z. The quantities T_0, Γ, Γ_a, and hence n, are assumed to be given. Then, V is given some arbitrary values ranging between 0.2×10^3 and 26×10^3 cm. sec.$^{-1}$ and the corresponding values of Z are found. The zeroes of Z are then found, at the places where there is a change of sign, by triple interpolation.

In the numerical example considered here the temperature of the tropopause T_0 was assumed to be 216°K., the height of the tropopause 11.594 km., and the lapse rate Γ was taken as 4.88°C./km. which is half the adiabatic lapse rate Γ_a. The value of the index n that corresponds to these values is 6. With these values the eigenvalues of V were found to get more crowded as the magnitude of V decreases. This is especially true between 10 m. sec.$^{-1}$ and 2 m. sec.$^{-1}$ At lower values one may be justified in considering them as if they constitute a continuous spectrum. In particular it was found that the following are eigenvalues of V: 1.01×10^3, 2.018×10^3, and 5.096×10^3 cm. sec.$^{-1}$ In the discussion to follow, these values are rounded out to 1, 2, 5 $\times 10^3$ respectively.

6. NUMERICAL DETERMINATION OF THE ELEVATIONS OF CLOUD STRATA

Equation (42) may now be used to determine the heights of the nodes and antinodes of the function ϕ, corresponding to any selected eigenvalue of V. These nodes and antinodes are the same for the vertical velocity w as has already been stated.

In the example worked out here the same numerical values have been assumed as in the last section. Figures 2a, b, and c are plots of ϕ against the elevation $z+h$ for the phase velocities 10 m. sec.$^{-1}$, 20 m. sec.$^{-1}$, and 50 m. sec.$^{-1}$, respectively. It may be seen from these figures that the troposphere is divided into distinct regions of vertical activity separated by nodal planes. Thus when $V=10$ m. sec.$^{-1}$, figure 2a provides for three regions with upward vertical velocities where cloud strata may form. The antinodes of these regions occur at the elevations 2, 6, and 10 km. All of these lie within the observed altitudes of low, medium, and high clouds respectively.

At the faster phase velocity of 20 m. sec.$^{-1}$, figure 2b shows that there can be only two regions with upward vertical velocities, hence only two cloud strata may form at this speed. At the still greater velocity of 50 m. sec.$^{-1}$, figure 2c shows that only one stratum may form.

It thus appears that, according to the mechanism proposed in the present communication, the number of cloud strata depends principally upon the phase velocity of the waves excited in the atmosphere. This number increases as the phase velocity decreases.

In order to test the effect of the lapse rate upon the possible number of strata the same problem was solved numerically for a lapse rate of 6.21° C./km. The eigenvalues of V at the lower end of the spectrum are not

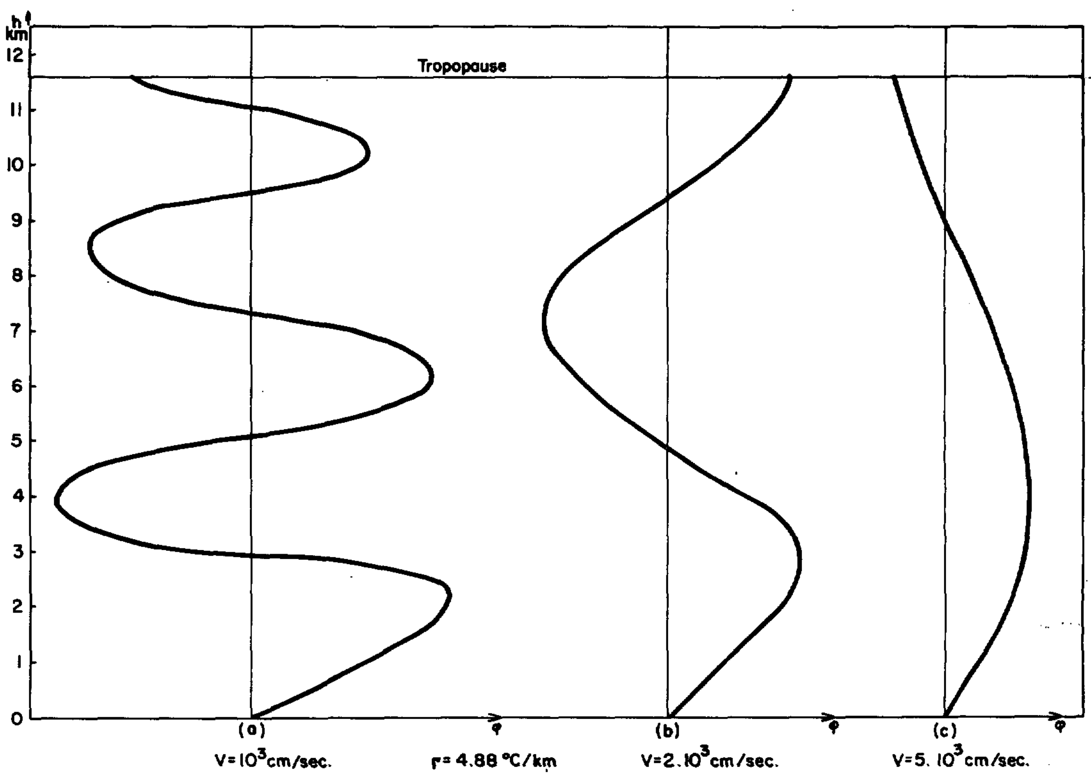

FIGURE 2.—A plot of the function ϕ against elevation in an atmosphere having a constant lapse rate $\Gamma=4.88°$ C./km. The temperature of the tropopause is assumed to be 216° K., and its height 11.594 km. Curve (a) corresponds to $V=10^3$ cm./sec., (b) corresponds to $V=2\times10^3$ cm./sec., and (c) corresponds to $V=5\times10^3$ km./sec.

changed appreciably for this case, and the same rounded values of 10, 20, 50 m. sec.$^{-1}$ were used for the sake of comparison. Figures 3a, b, and c are the result. It may be seen from these figures that the altitudes of the possible strata occur at about the same values as in the previous case, except that these altitudes are now slightly shifted upward. It is therefore surmised that the conclusion drawn in the last paragraph remains valid, namely that the number of strata is mainly a function of the phase velocity.

7. WAVES OF THE SECOND KIND

In this case the assumption is made that $\sigma^2 \approx -f^2$. For the isothermal stratosphere equation (18) takes the form

$$c_0^2\chi_{zz}-\gamma g\chi_z-\frac{g^2k^2}{f^2}(\gamma-1)\chi=0. \qquad (46)$$

The solution for this equation is:

$$\chi=D_2e^{rz} \qquad (47)$$

where

$$r=\frac{\gamma g}{2c_0^2}\left[1-\left(1+\frac{4c_0^2k^2}{f^2}(\gamma-1)\right)^{1/2}\right] \qquad (48)$$

and D_2 is a constant of integration.

Because the radical is always real, the negative sign of the ambiguity has been chosen in order to make the variables approach zero at great heights.

Upon substituting from equation (47) in (15) the following value is obtained for the vertical velocity w:

$$w=D_2\left[\frac{k^2c_0^2+\gamma f^2}{gk^2}-\frac{f^2c_0^2}{g^2k^2}r\right]e^{rz}. \qquad (49)$$

It is obvious from equation (49) that no nodal planes can exist in this case since w is not a periodic function.

For the troposphere, equation (22) takes the form

$$\zeta\chi_{\zeta\zeta}+(n+2)\chi_\zeta-\mu\chi=0 \qquad (50)$$

where

$$\mu=\frac{gk^2}{f^2}\left(\frac{\Gamma_a}{\Gamma}-1\right). \qquad (51)$$

Upon making the following changes in variables

$$\left.\begin{aligned}\chi&=\zeta^{-\frac{n+1}{2}}\psi\\ \text{and}\qquad \lambda^2&=4\mu\zeta\end{aligned}\right\} \qquad (52)$$

equation (50) takes the form:

$$\psi''+\frac{1}{\lambda}\psi'-\left[1+\frac{(n+1)^2}{\lambda^2}\right]\psi=0 \qquad (53)$$

where primes indicate differentiation with respect to λ. The solution of equation (53) is:

$$\psi=A_2I_{n+1}(\lambda)+B_2K_{n+1}(\lambda) \qquad (54)$$

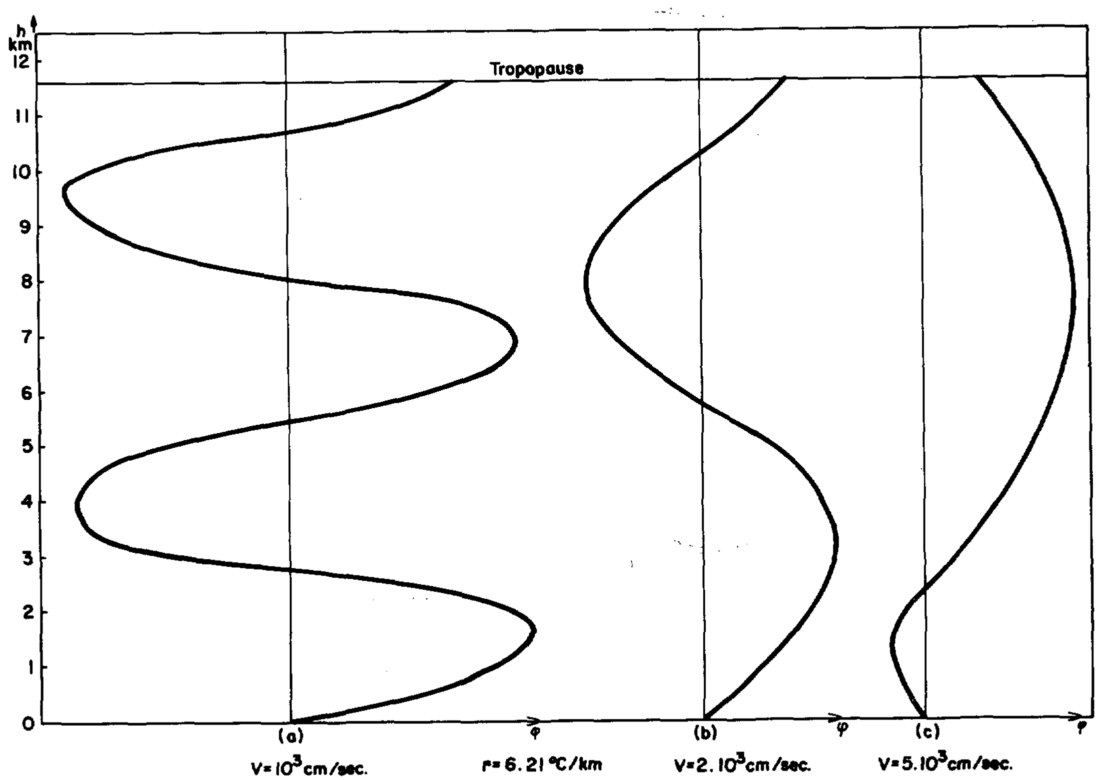

FIGURE 3.—Same as figure 2 with $\Gamma=6.21°$ C./km.

where I and K are the modified Bessel functions of the two kinds.

Upon substituting from equation (54) in (52) then using equation (15) and the boundary condition (37), the following values for χ and w are obtained:

$$\chi=(4\mu)^{\frac{n+1}{2}}\lambda^{-(n+1)}[A_2I_{n+1}(\lambda)+B_2K_{n+1}(\lambda)] \tag{55}$$

and

$$w=A_2\frac{\gamma f^2}{2(n+1)gk^2}(4\mu)^{\frac{n+1}{2}}\lambda^{-n}\left\{\left[I_n(\lambda)+\frac{gk^2}{2\mu f^2}\lambda I_{n+1}(\lambda)\right]-C_2\left[K_n(\lambda)-\frac{gk^2}{2\mu f^2}\lambda K_{n+1}(\lambda)\right]\right\} \tag{56}$$

where

$$C_2=\frac{B_2}{A_2}=\frac{\left[I_n(\lambda_{-h})+\frac{gk^2}{2\mu f^2}\lambda_{-h}I_{n+1}(\lambda_{-h})\right]}{\left[K_n(\lambda_{-h})-\frac{gk^2}{2\mu f^2}\lambda_{-h}K_{n+1}(\lambda_{-h})\right]}. \tag{57}$$

The subscript $-h$ indicates that the value at $z=-h$ should be taken.

The condition for nodal planes in w is, from equation (56):

$$\left[I_n(\lambda)+\frac{gk^2}{2\mu f^2}\lambda I_{n+1}(\lambda)\right]-C_2\left[K_n(\lambda)-\frac{gk^2}{2\mu f^2}\lambda K_{n+1}(\lambda)\right]=0. \tag{58}$$

Because the functions I and K are monotones, behaving like exponentials, no real values of λ can satisfy equation (58) except λ_{-h}, which is the ground level. Hence no nodal planes above the ground can exist in waves of the second kind. It is therefore felt unnecessary to formulate the eigenvalue problem for this case. The conclusion arrived at here means, in turn, that long waves belonging to the second kind cannot give rise to stratified cloud layers of the nature stipulated. Any stratification that may be observed in association with these long waves must be explained by other mechanisms, and not by simple free gravitational oscillations.

8. CONCLUSIONS AND FURTHER REMARKS

On the basis of the above-cited analysis the following conclusions and remarks may be made:

(1) Atmospheric oscillations whose frequencies are greater than the Coriolis parameter may be associated with the formation of cloud strata. In these short-wave oscillations gravity is the main controlling force. The heights of the strata that may be associated with these oscillations in an isothermal stratosphere are in qualitative agreement with the observed heights of some cloud formations that appear in the upper atmosphere.

Short-wave oscillations may also account for the stratifications of clouds that are occasionally observed in a stable troposphere. In this case, it is found that fast moving waves of phase velocities 50 m. sec.$^{-1}$ or more may give rise to one stratum only. Slower waves may produce more strata, so that at a velocity of the order 10 m. sec.$^{-1}$ there may be three strata generated by this mechanism. The elevations of these strata fall within the limits of the observed elevations for tropospheric clouds.

(2) Variation of the lapse rate of the troposphere does not produce appreciable changes in the qualitative results, although it may result in slight changes in the actual heights of the most favorable places for cloud formation.

(3) Oscillations whose frequencies are less than the Coriolis parameter, and which belong to long wavelengths, are not capable of producing cloud strata, either in the isothermal stratosphere or in the troposphere. In this case a vertical column of the atmosphere vibrates in phase with itself so that no nodal planes can exist. Cloud stratification that may be observed in association with these waves must be explained by other mechanisms, such as the bodily lifting of air masses over frontal surfaces.

(4) In order to simplify the mathematical analysis the hydrostatic assumption has been made in the present treatment. This assumption will be justified in the Appendix. However, this assumption imposes certain limitations on the phase velocities which may be considered. The discussion therefore was not carried out for velocities less than 10 m. sec.$^{-1}$

(5) In the present model the undisturbed atmosphere was assumed to be stagnant. A basic current with vertical wind shear may result in some significant modifications (see Sekera [8]). This however, was not attempted in the present article since the main object has been to demonstrate the feasibility of the basic mechanism suggested here.

APPENDIX

ON THE VALIDITY OF THE HYDROSTATIC ASSUMPTION

In order to test the validity of the hydrostatic assumption, and to bring out the nature of the approximations implied by introducing it, it may be instructive to derive the same basic equations without making use of this assumption. Equations (5)–(10) will then be

$$u_t-fv=-\frac{1}{\rho_0}p_x \tag{5a}$$

$$v_t+fu=0 \tag{6a}$$

$$w_t=-\frac{1}{\rho_0}p_z-g\rho \tag{7a}$$

$$\rho_t+w\rho_{0z}+\rho_0\chi=0 \tag{8a}$$

$$c^2(\rho_t+w\rho_{0z})=p_t-wg\rho_0 \tag{9a}$$

$$\chi=u_x+w_z \tag{10a}$$

The only difference between these and the corresponding equations (5)–(10) appears in (7a) which is now written to include the vertical accelerations.

Upon eliminating v, p, ρ, and u the following equations are obtained

$$(\nu^2\sigma^2-k^2g^2)w=g[\gamma\sigma^2-k^2c^2]\chi-c^2\sigma^2(d\chi/dz) \qquad (59)$$

and

$$c^2\frac{d^2\chi}{dz^2}+\left(\frac{dc^2}{dz}-g\gamma\right)\frac{d\chi}{dz}+\left\{\frac{gk^2}{\sigma^2}\left[\frac{dc^2}{dz}+g(\gamma-1)\right]+\nu^2\left(1-\frac{k^2c^2}{\sigma^2}\right)\right\}\chi=0. \qquad (60)$$

Comparison of these two equations with the corresponding equations (15) and (16) shows that (a) in equation (59) there is an additional term, $\nu^2\sigma^2$, in the coefficient of w, and (b) in equation (60) there is an additional term, $\nu^2\left(1-\frac{k^2c^2}{\sigma^2}\right)$, in the coefficient of χ. It will now be shown that these terms are negligible in comparison with the retained terms, for the range of waves we are considering.

For the waves of the first kind $\sigma^2\approx\nu^2$ and therefore $\nu^2\sigma^2\approx\nu^4$. For the waves of the second kind $\sigma^2\approx -f^2$, and in the order of magnitudes considered here $f^2<\nu^2$. Hence, if it could be shown that $\nu^4<<k^2g^2$, it follows that this approximation holds much better for $\sigma^2\nu^2$ in general.

The orders of magnitude of the parameters involved in the present treatment are: $\nu=O$ (10^{-3} sec.$^{-1}$), $k=O$ (10^{-6} cm.$^{-1}$). Hence, $\nu^4=O$ (10^{-12}) and $k^2g^2=O$ (10^{-6}). It is therefore clear that k^2g^2 is six orders of magnitude greater than $\nu^2\sigma^2$ which makes the latter term negligible in comparison with the first. In general if $\nu^4<k^2g^2$ then $\nu V<g$.

For $V=O$ (10^3), ν must be of the order of 10^{-1} or less to make the neglected term two orders or more smaller than the retained term. This is always true in the range considered here.

Next consider the relative magnitudes of the terms in the coefficient of χ of equation (60). The first term is $(g^2k^2/\sigma^2)(\gamma-1)$. The order of magnitude of this term is the same as that of g^2k^2/ν^2 or g^2/V^2, which is, for $V=10^3$, of the order of $10^0=1$. Whereas the term $\nu^2(1-k^2c^2/\nu^2)=\nu^2-k^2c^2$. The first part, ν^2, is always smaller than 1. The second, i.e., $k^2c^2=O$ ($10^{-12}\cdot 10^9$)$=O$ (10^{-3}), which is three orders of magnitude smaller than the retained term; hence it may be neglected. In general if $k^2c^2<g^2/V^2$, then $k^2<10^{-3}/V^2$.

For V of $O(10^3)$, k must be smaller than 3×10^{-4}. In other words this approximation may be made for wavelengths of the order of 1 m. even if the velocity of these waves remains as high as 10 m. sec.$^{-1}$ It is therefore clearly demonstrated that the hydrostatic assumption is more than justified for the range of values of interest in the present investigation. In addition, the introduction of the hydrostatic assumption has the advantage of simplifying the mathematical analysis considerably.

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