WAVE MOTION AT THE SURFACE OF A CURRENT WHICH HAS AN EXPONENTIAL DISTRIBUTION OF VORTICITY

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Introduction. The study of waves that are propagated at the surface of a current is greatly complicated by different factors. Among those factors one might mention the earth's rotation, turbulence, viscosity, friction with outside bodies, and the vorticity distribution in the underlying basic current. Various attempts were made to investigate the influence of these factors. In these investigations, each factor might have been taken separately, or two or more of them might have been assumed to work together. The problem of the influence of vorticity distribution, however, has been restricted to some idealized pictures. The present paper might well be classified in this category. For in the present paper, the vorticity distribution is assumed to follow an exponential law. The vorticity is assumed to vanish at a great depth and to attain its maximum magnitude at the free surface. This, however, is not totally without a justifying reason. For, through the work of Ekman,[1] it is known that the surface oceanic currents follow, approximately, a spiral law. So that

$$U = V_0 e^{\alpha z} \cos\left(\frac{\pi}{4} + \alpha z\right)$$

and

$$V = V_0 e^{\alpha z} \sin\left(\frac{\pi}{4} + \alpha z\right) \qquad (1)$$

where U and V are the components of the velocity in the x- and y-directions respectively; the x-y plane being horizontal and in the undisturbed free surface, the z-axis being taken vertical (positive upwards). α is a constant which has the value

$$\alpha = +\sqrt{\frac{\rho\omega \sin\theta}{\mu}} \qquad (2)$$

where ρ is the constant density, ω is the angular velocity of the earth's rotation, θ is the latitude, and μ is the coefficient of virtual viscosity. In EQUATION 1, V_0 is the absolute velocity of the water at the surface which has the value

$$V_0 = \frac{T}{\mu\alpha\sqrt{2}} \qquad (3)$$

where T is the tangential pressure of the wind on the free surface. This tangential pressure is directed along the positive axis of y. The surface velocity of the water, V_0, is related to the wind velocity U' by the following equation, which was derived by Ekman (see Sverdrup[2]),

$$\frac{V_0}{U'} = \frac{0.0127}{\sqrt{\sin \theta}}. \qquad (3.1)$$

Furthermore, it was pointed out by Ekman[1] that, according to EQUATION 1, the velocity vector turns around with depth, and at a certain depth D the direction becomes opposite to the direction at the surface. The depth D was called by Ekman the "Depth of Wind currents". D is related to α by the equation

$$D = \frac{\pi}{\alpha}. \qquad (2.1)$$

Ekman's computations, however, show that by the time the depth D is reached the speed of the current is reduced to a small fraction of the speed at the surface.

In view of the preceding discussion and in view of the fact that waves do occur at the surface of ocean currents that are driven by winds, it seems interesting to investigate wave motion in these currents. The mathematical solution of this problem, in its general form, is rendered difficult because of several factors. First, the magnitude of the current speed decreases exponentially with depth, as is easily seen from EQUATION 1. This current distribution imposes a vorticity distribution. The subsequent wave motion thereby inherits a rotational motion, and the usual assumption of a velocity potential is therefore inadequate. A second factor is that the current changes direction. The motion, therefore, has to be treated in three dimensions. A third complicating factor is that with such a current distribution, like the one we are dealing with, the flow is not laminar. Turbulence becomes an important factor and it plays a role that may even overshadow that of gravity.

In the present paper, an attempt is made to deal with some aspects of this problem. To simplify matters only one of the above-mentioned factors is considered; namely, the effect of the speed distribution. The assumption is made that the effect of turbulence is to build up the basic current in a gradual manner. It is then assumed that the effect of turbulence on perturbation motion may be neglected. The flow may then be considered as laminar and free from viscosity. It is further assumed that the effect of the earth's rotation on the perturbation motion may also be neglected. Furthermore, the rotation of the velocity vector with depth is neglected and the current is assumed to be in the same direction throughout. The motion is therefore reduced to two dimensions.

The magnitude of the basic current, in the absence of vertical motion, is, from EQUATION 1,

$$U = V_0 e^{\alpha z}; \qquad (4)$$

and, if the variation of the direction is neglected, the x-axis taken to coincide with the direction of the current, and the z-axis vertical and pointing upwards as before, the equations of the perturbation motion may be written as follows

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)u + w\frac{\partial U}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \cdots \text{(a)} \\ \text{and} \quad \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)w &= -\frac{1}{\rho}\frac{\partial p}{\partial z} \cdots \text{(b)} \end{aligned}\right] \tag{5}$$

The linearized equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{6}$$

IN EQUATIONS 5 and 6, u and w are the components of the perturbation velocity in the directions x and z respectively, p is the perturbation pressure, and ρ is the constant density.

The equation of continuity suggests the use of a stream function ψ such that

$$\left.\begin{aligned} u &= -\frac{\partial \psi}{\partial z} \\ \text{and} \quad w &= \frac{\partial \psi}{\partial x} \end{aligned}\right] \tag{7}$$

Upon differentiating (5a) partially with respect to z and (5b) partially with respect to x and subtracting, the following equation is obtained

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\xi + w\frac{\partial^2 U}{\partial z^2} = 0 \tag{8}$$

where $\xi = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$ is the vorticity. Upon substituting in (8) from (7) the former becomes

$$\left(\frac{\partial}{\partial x} + U\frac{\partial}{\partial t}\right)\nabla^2\psi - \frac{\partial \psi}{\partial x}\frac{\partial^2 U}{\partial z^2} = 0 \tag{9}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.$$

It is easily verified that EQUATION 9 agrees with the equation found by Rayleigh[3] and later by G. I. Taylor.[4]

The solution of (9) may be assumed to have the form

$$\psi = \varphi(z)e^{i(mx-\sigma t)} \tag{10}$$

where φ is a function of z only. Using the value of U as given in (4) and substituting from (10) in EQUATION 9, the following equation for φ is found

$$\varphi'' - \left(m^2 + \frac{\alpha^2 mU}{mU - \sigma}\right)\varphi = 0 \tag{11}$$

where the primes indicate differentiation with respect to z. The substitution

$$s = mU \tag{12}$$

transforms this equation to the following form

$$s^2 \frac{d^2\varphi}{ds^2} + s\frac{d\varphi}{ds} - \left(\frac{m^2}{\alpha^2} + \frac{s}{s - \sigma}\right)\varphi = 0 \tag{13}$$

where now the independent variable is s. If, in EQUATION 13, the following substitution is made

$$\varphi = s^n v \tag{14}$$

where $n = \dfrac{m}{\alpha}$, the following equation for v is found

$$s\frac{d^2v}{ds^2} + (2n + 1)\frac{dv}{ds} - \frac{v}{s - \sigma} = 0 \tag{15}$$

EQUATION 15 is a Frobenius type equation. It has singularities at $s = 0$, σ, $\pm\infty$. There are, therefore, three types of solutions. The solutions around $\pm\infty$ are, however, irrelevant to the physical problem which we are dealing with. This is because s is usually a small number and does not approach infinity. Hence we proceed to derive the other two solutions.

The solution around 0.

Upon setting

$$v = s^j \sum_{i=0}^{\infty} a_i s^i \tag{16}$$

and substituting in (15), the indicial equation is found to be

$$\sigma j(j + 2n) = 0. \tag{17}$$

Thus $j = 0$, or $-2n$. Taking $j = 0$ the general term is

$$a_r s^r = \frac{(\Gamma k)(k-1)(2k+1)(3k+5)\cdots(r-1)[(k+r-2)-1]}{\sigma a^r(r!)\Gamma(k+r-1)} a_0\left(\frac{s}{\sigma}\right)^r \cdots \tag{18}$$

where Γ is the gamma function and $k = 2n + 1$, and the solution is

$$v_1 = \frac{1}{\sigma(2n+1)}\left[1 + \frac{2n}{2(n+1)}\left(\frac{s}{\sigma}\right) + \frac{n(4n+3)}{3!(n+1)(2n+3)}\left(\frac{s}{\sigma}\right)^2 + \cdots\right] \tag{19}$$

and taking $j = -2n$, the solution may be written in the following concise form

$$v_2 = s^{-2n}K(s) \tag{20}$$

where $K(s)$ is an ascending power series in s. The explicit form of $K(s)$ need not be written here for reasons that will appear immediately. Hence the complete solution for EQUATION 13 is

$$\varphi = C_1v_1 + C_2v_2 \tag{21}$$

where C_2 and C_1 are arbitrary constants. The boundary condition that, at $z = -\infty$, the motion has to remain finite gives

$$C_2 = 0 \qquad \text{and} \qquad C_1 = C.$$

Hence the complete solution for our purpose is

$$\varphi = CS_1(s) \tag{22}$$

where

$$S_1(s) \equiv \frac{s^n}{\sigma(2n+1)}\left[1 + \frac{n}{2!(n+1)}\left(\frac{s}{\sigma}\right) + \frac{n(4n+3)}{3!(n+1)(2n+3)}\left(\frac{s}{\sigma}\right)^2 + \cdots\right]. \tag{23}$$

From (10) it follows that

$$\psi = CS_1(s)e^{i(mx-\sigma t)} \tag{24}$$

and from (7) it follows that

$$\left.\begin{aligned} w &= imCS_1(s)e^{i(mx-\sigma t)} \cdots \text{(a)} \\ \text{and} \quad u &= -\alpha CS_2(s)e^{i(mx-\sigma t)} \cdots \text{(b)} \end{aligned}\right] \tag{25}$$

where

$$S_2(s) \equiv \frac{s^n}{\sigma(2n+1)}\left[n + \frac{n(n+1)}{2!(n+1)}\left(\frac{s}{\sigma}\right) + \frac{n(n+2)(4n+3)}{3!(n+1)(2n+3)}\left(\frac{s}{\sigma}\right)^2 + \cdots\right]. \tag{26}$$

To find p the assumption is made that

$$\frac{P}{\rho} = D(z)e^{i(mx-\sigma t)} \tag{27}$$

where $D(z)$ is a function of z only. If this value is substituted in (5), and the EQUATIONS 24, 25, and 26 are used, it follows that

$$\frac{P}{\rho} = -\left[\frac{\sigma - mU}{m}\alpha S_2(s) + \alpha U S_1(s)\right] Ce^{i(mx-\sigma t)}. \tag{28}$$

The boundary condition at the free surface is (see Haurwitz[4])

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{p}{\rho} - wg = 0; \quad \text{at} \quad z = 0 \tag{29}$$

where g is the acceleration of gravity.

$$\left.\begin{aligned} &\text{Now at} \quad z = 0 \\ &\qquad s_0 = mV_0, \\ &\qquad w_0 = imCS_{01}(s_0) \\ &\text{and} \quad \frac{p_0}{\rho} = -(c - v_0)C\alpha S_{02} - C\alpha V_0 S_{01} \end{aligned}\right] \tag{30}$$

where the common exponential factor is understood. In (10), and also in (30), $m = 2\pi\lambda^{-1}$, where λ is the wave length, and σ is the frequence so that

$$c = \frac{\sigma}{m}$$

is the phase velocity of the sinusoidal waves which travel at the surface. Upon substituting from (30) in (29), the following frequency equation is found

$$(c - V_0)^2\frac{S_{02}}{S_{01}} + V_0(c - V_0) - \frac{g}{\alpha} = 0. \tag{31}$$

This equation can also be written in the form

$$c = V_0 - \tfrac{1}{2}V_0\frac{S_{01}}{S_{02}} \pm \sqrt{\left(\tfrac{1}{2}V_0\frac{S_{01}}{S_{02}}\right)^2 + \frac{g}{\alpha}\frac{S_{01}}{S_{02}}}. \tag{32}$$

This equation is not readily susceptible to computation. This is because both S_{01} and S_{02} involve the unknown quantity c in the form of sums of terms in the ascending powers of this quantity. This may be seen readily from the identities (23) and (26). Nevertheless, a preliminary discussion of its implications may be given.

In the first place, it is easily seen from the identities (23) and (26) that the limit of the ratio $\frac{S_{02}}{S_{01}}$ approaches n as s approaches zero. That is

$$\lim_{s\to 0}\left(\frac{S_{02}}{S_{01}}\right)\to n.$$

It is also obvious that s goes to zero if V_0 goes to zero. Thus

$$\lim_{V_0\to 0}\left(\frac{S_{02}}{S_{01}}\right)\to n. \tag{33}$$

But, either from (31), or from (32), it is seen that

when $$V_0 \to 0$$

$$c \to \pm\sqrt{\frac{g}{\alpha}\lim_{V_0\to 0}\left(\frac{S_{01}}{S_{02}}\right)} \tag{34}$$

or $$c \to \pm\sqrt{\frac{g}{\alpha}\frac{1}{n}}$$

And since $n = \frac{m}{\alpha}$, (34) reduces to

$$c = \pm\sqrt{\frac{g}{m}}, \qquad \text{when} \quad V_0 = 0, \tag{35}$$

which is the Stokes's velocity (see Lamb).[6] This result is in agreement with what is expected; since, in virtue of EQUATION 4, the basic current vanishes when the surface current vanishes. And if there is no basic current, the phase velocity follows the Stokes's formula for the first degree of approximation. It is also to be noticed from the identities (23) and (26) that the limit given in (33) is valid if V_0 is small compared with c. Thus, (35) can be put in the form

$$c \xrightarrow[(V_0/c)\to 0]{} \pm\sqrt{\frac{g}{m}}, \tag{36}$$

which shows that Stokes's formula is approached if the surface current is small compared with the wave velocity.

It appears from EQUATION 32 that there are two factors affecting the wave propagation in the case under consideration. These are gravity and the current shear. Because the shear is originally produced by turbulence, this latter factor will be called the turbulence factor. In EQUATION 32, the gravity factor is expressed by the term

$$\frac{g}{\alpha}\frac{S_{01}}{S_{02}},$$

which appears under the radical. The turbulence factor is expressed by the term

$$\tfrac{1}{2}V_0 \frac{S_{01}}{S_{02}},$$

which appears both outside and inside the radical. Now, since the system of axes is so chosen as to make V_0 always positive, and since the ratio $\frac{S_{01}}{S_{02}}$ is always positive, it follows that

$$\tfrac{1}{2}V_0 \frac{S_{01}}{S_{02}} > 0.$$

Therefore

$$\left|\sqrt{\left(\tfrac{1}{2}V_0 \frac{S_{01}}{S_{02}}\right)^2 + \frac{g}{\alpha}\frac{S_{01}}{S_{02}}}\right| > \left|\tfrac{1}{2}V_0 \frac{S_{01}}{S_{02}}\right|. \tag{37}$$

This shows that nothing could be said about the relative magnitude of the speed of waves travelling in the direction of the current to the speed of waves travelling against the current. The two speeds may or may not be the same.

Again, if the absolute magnitude of the radical is taken, it is seen that

$$\left|\sqrt{\left(\tfrac{1}{2}V_0 \frac{S_{01}}{S_{02}}\right)^2 + \frac{g}{\alpha}\frac{S_{01}}{S_{02}}} - \left(\tfrac{1}{2}V_0 \frac{S_{01}}{S_{02}}\right)\right| < \left|\sqrt{\frac{g}{\alpha}\frac{S_{01}}{S_{02}}}\right|. \tag{38}$$

This shows that the total effect of turbulence and gravity is to give the waves a velocity which is smaller than that provided by gravity alone. Turbulence is, therefore, a retarding factor. This, again is expected, since turbulence is, in a way, a frictional force like viscosity.

It has already been remarked that neither EQUATION 32 nor EQUATION 31 is readily susceptible to computation. Therefore the numerical computations from these equations has to be done by some method of approximation. The method followed here is that of successive approximations. From the identities (23) and (26) it follows that the ratio $\frac{S_{02}}{S_{01}}$ has the value

$$\frac{S_{02}}{S_{01}} = n\left[1 + \frac{1}{2(n+1)}\left(\frac{V_0}{c}\right) + \frac{10n^2 + 19n + 3}{12(n+1)^2(2n+3)}\left(\frac{V_0}{c}\right)^2 + \cdots\right] \tag{39}$$

and from EQUATION 31 it follows that

$$m = \frac{ng}{(c - V_0)^2 \frac{S_{02}}{S_{01}} + V_0(c - V_0)}. \tag{40}$$

The first approximation would be to put

$$\frac{S_{02}}{S_{01}} = n_1 = \frac{m_1}{\alpha}.$$

EQUATION 40 then gives

$$m_1 = \frac{g - \alpha V_0(c - V_0)}{(c - V_0)^2}. \tag{40.1}$$

This value for m may then be used in the EQUATION 39 and a second approximation may be found. The second approximation may again be used to find a third approximation, and so on.

It may be remarked that the series $\frac{S_{02}}{S_{01}}$ given in (39) converges rapidly

TABLE 1

$V_0 = 15$ cm/sec., $\alpha = 10^{-3}$ c.g.s., $g = 980$ c.g.s.

c m/sec.	First approx.		Second approx.		Stokes values	
	m_1	λ ms.	m_2	λ m.	m_s	λ_sm.
1.00	$1.35\cdot10^{-1}$	0.465			1.355×10^{-1}	0.464
2	$2.86\cdot10^{-2}$	2.20			2.86×10^{-2}	2.19
3	$1.205\cdot10^{-2}$	5.12	$1.06\cdot10^{-2}$	5.92	1.206	5.21
4	0.657	9.55			0.661	9.50
5	0.414	15.18	0.423	14.81	0.416	15.11
6	$2.83\cdot10^{-3}$	22.09			2.85×10^{-3}	22.01
7	2.07	33.20			2.09	30.05
8	1.565	40.02			1.585	39.7
9	1.235	50.08			1.25	50.03
10	0.994	63.10	$0.99\cdot10^{-3}$	63.4	1.01	62.02
11	$8.85\cdot10^{-4}$	71.00			8.97×10^{-4}	69.9
12	6.82	92.05			6.95	90.4
13	5.82	107.80			5.93	106.0
14	5.01	125.10			5.10	123.3
15	4.32	145.20	$4.3\cdot10^{-4}$	146.0	4.42	142.0
16	3.81×10^{-4}	164.90			3.905×10^{-4}	160.8
17	3.36	187.00			3.45	182.0
18	3.02	208.60			3.08	202.0
19	2.68	234.10			2.76	227.8
20	2.41	260.50	$2.41\cdot10^{-4}$	260.50	2.48	253.0
25	1.54	407.00			1.605	391
30	1.05	598.00			1.1	571
35	0.761	824.00			0.804	781
40	0.572	1099.00			0.616	1021

and uniformly for values of $\frac{V_0}{c}$ that are less than unity. The fact that the series converges rapidly enables us to neglect terms of third and higher order, and thus the numerical computation is made easier.

TABLES 1 and 2 give the first and second approximations as computed by the method of successive approximations.

From EQUATION 3.1 it can be seen that a surface velocity of 15 cm/sec is caused by a wind of about 10 m/sec, at latitude 43°. The value of α was computed from the corresponding value of D as given by Sverdrup[3] (page 23). The Stokes's values were computed from the equation

$$m = \frac{g}{(c - V_0)^2}.$$

FIGURE 1 is a plot of these values.

It can be seen from these tables that the second approximation gives for m values that are slightly higher than those given by the first approximation. The difference, however, is small and the correction involved is of the order of 3 per thousand.

FIGURE 2 is a plot based on TABLE 2.

It can also be seen either from FIGURES 1 and 2 or from TABLES 1 and 2 that Stokes's values for the velocity are higher than the values given by (40). This is in agreement with the discussion given in this paper.

TABLE 2

$V_0 = 10$ cm/sec., $\alpha = 10^{-4}$ c.g.s., $g = 980$ c.g.s.

c m/ sec.	*First approx.*		*Second approx.*		*Stokes values*	
	m_1	λ ms.	m_2	λ m.	m_s	$\lambda_s m.$
1	1.555	0	∞	0	∞	0.0
2	2.555×10^{-2}	0.641			9.8×10^{-2}	1.00
3	2.555	2.562			2.45	2.64
4	0.586	5.78			1.09	5.56
5	0.50×10^{-3}	10.30	6.08×10^{-3}	10.42	0.61×10^{-3}	10.75
6	2.59	16.15			3.92	16.28
7	2.405	23.09			2.72	23.00
8	1.485	31.60			2.0	31.42
9	1.513	41.04			1.53	41.00
10	1.20	52.30	1.18×10^{-3}	53.60	1.21	51.80
11	9.7×10^{-4}	64.80			0.98	64.00
12	8.0 ·	78.50			0.81	77.25
13	6.22	100.80			0.68	92.50
14	5.72	109.90			0.58	108.20
15	4.92	127.60	4.91×10^{-4}	127.80	5.0×10^{-4}	125.50
16	4.34	144.90			4.35	144.10
17	3.77	167.00			3.83	163.80
18	3.33	188.50			3.39	185.00
19	2.97	210.80			3.02	207.90
20	2.66	236.00	2.63×10^{-4}	238.5	2.70	232.30
25	1.66	378.00	1.65	380.4	1.71	367.8
30	1.13	561.00	1.12	560.8	1.66	541.0
35	0.8	764.00			0.848	742.1
40	0.618	1015.00			0.643	979.0

The solution around σ.

The previous discussion was based on the assumption that s is very small; that is, that the ratio $\frac{V_0}{c}$ is in the neighborhood of zero. It is thought that this has some physical significance, since the wave velocity is usually much greater than the velocity of the basic current. However, a solution for EQUATION 15 may be found for values of s that are in the neighborhood of σ; that is, for values of $\frac{V_0}{c}$ near unity. This does not seem to have much physical bearing since c is usually much greater than V_0. Nevertheless, the solution for this case will here be given for the sake of completeness.

If in EQUATION 15 the substitution

$$s - \sigma = s_1$$

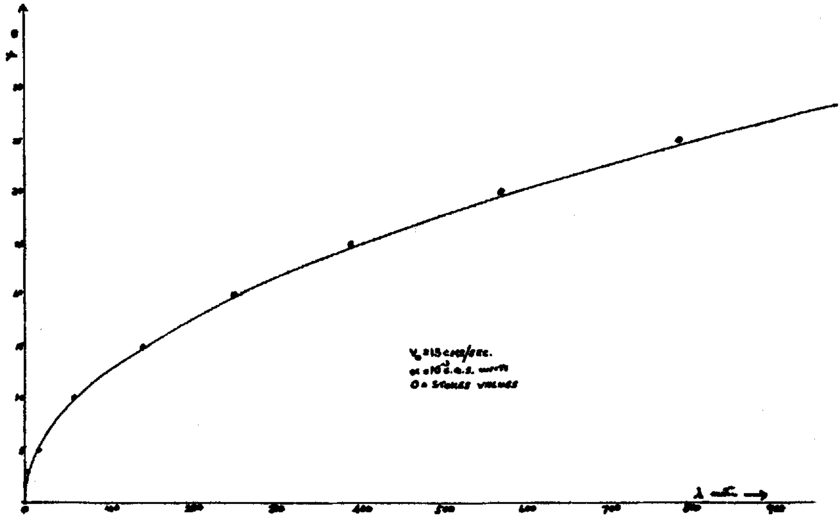

FIGURE 1.

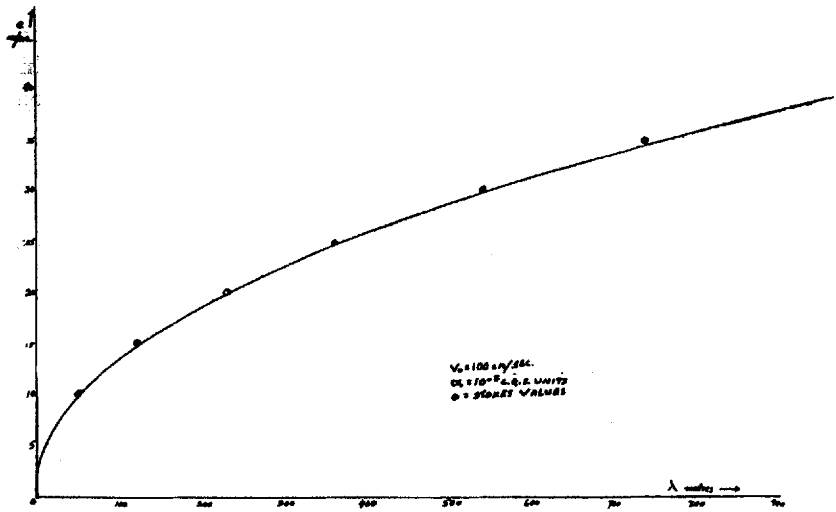

FIGURE 2.

is made, the equation takes the form

$$(s_1 + \sigma)s_1 \frac{d^2 v}{ds_1^2} + ks_1 \frac{dv}{ds_1} - v = 0, \qquad k = 2n + 1. \tag{41}$$

The solution for (41) may be assumed to have the form

$$v = s_1^j(a_0 + a_1 s_1 + a_2 s_1^2 + \cdots a_r s^r + \cdots).$$

The indicial equation is

$$j(j-1)=0$$

and the $r+1$ th term is

$$a_{r+1}s_1^{r+1} = \frac{1-(r+j)(r+j-1-k)}{\sigma(r+j)(r+j+1)}a_r s^r.$$

Taking $j=0$, this becomes

$$a_{r+1}s^{r+1} = \frac{1-r(r-k-1)}{\sigma r(r+1)}a_r s^r$$

and the series solution in terms of the original variable s is

$$v = C(s-\sigma)H_1 \tag{42}$$

where

$$H_1 = \left[1+\frac{1+k}{2}\left(\frac{s}{\sigma}-1\right) + \frac{(1+k)(2k-1)}{2^2\cdot 3}\left(\frac{s}{\sigma}-1\right)^2+\cdots\right]. \tag{43}$$

Taking $j=1$, a solution identical to (43) is found. An additional logarithmic solution may then be written, but this is not necessary because this additional solution will vanish in virtue of the lower boundary condition. Hence the complete solution for our purpose is

$$\psi = Cs^n(s-\sigma)H_1 e^{i(mx-\sigma t)}$$

Hence

$$w = imCs^n(s-\sigma)H_1 e^{i(mx-\sigma t)} \tag{44}$$

and

$$u = [\alpha Cn(s-\sigma)s^n H_1 + \alpha Cs^{n+1}H_2]e^{i(mx-\sigma t)}$$

where

$$H_2 = 1+\frac{2(1+k)}{2\sigma}(s-\sigma)+3\frac{(1+k)(2k-1)}{2^2\cdot 3\cdot\sigma^2}(s-\sigma)^2+\cdots. \tag{45}$$

And from (5) it is found that

$$\frac{p}{\rho} = \left\{\alpha\frac{\sigma-s}{m}s^n[n(s-\sigma)H_1+sH_2] - \frac{\alpha s^{n+1}}{m}(s-\sigma)H_1\right\}Ce^{i(mx-\sigma t)}. \tag{46}$$

Upon substituting in the boundary condition for the upper free surface, which is given in EQUATION 29, the following frequency equation is found

$$\sigma - mV_0 = \frac{\alpha V_0}{2}\left(\frac{H_2}{H_1}-1\right) \pm \sqrt{\left(\frac{\alpha V_0}{2}\right)^2\left(\frac{H_2}{H_1}-1\right)^2+mg}. \tag{47}$$

It can be seen that, in (47), when

$$V_0 \to 0$$

$$\sigma \to \pm \sqrt{mg}.$$

Also when $V_0 \to c$, the ratio $\frac{H_2}{H_1} \to 1$, and

$$\sigma \to \pm \sqrt{mg}.$$

which is the value that one expects.

Stability of Surface Waves.

To investigate the stability of the waves under consideration it will be assumed that, in the undisturbed case, the air flows with a velocity which is constant with height. Let the component of this velocity in the direction of x be U'. It will further be assumed that the wave motion in the air is symmetrical with respect to the y-axis. The air will be considered as a homogeneous non-viscous fluid of constant density ρ'. The compressibility of the air is neglected because the speeds we are dealing with are much smaller than the speed of sound. Under these assumptions the equations of the perturbation motion in the air will be

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} + U'\frac{\partial}{\partial x}\right)u' &= -\frac{1}{\rho'}\frac{\partial p'}{\partial x} \quad \text{(a)} \\ \text{and} \quad \left(\frac{\partial}{\partial t} + U'\frac{\partial}{\partial x}\right)w' &= -\frac{1}{\rho'}\frac{\partial p'}{\partial z} \quad \text{(b)} \end{aligned}\right] \tag{48}$$

and the linearized equation of continuity is

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \tag{49}$$

where the primes refer to the upper fluid, which is the air.

Upon differentiating (48a) partially, with respect to z, and (48b), with respect to x, and subtracting, the following equation is obtained

$$\left(\frac{\partial}{\partial t} + U'\frac{\partial}{\partial x}\right)\left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right) = 0. \tag{48.1}$$

EQUATION 49 suggests the use of the stream function ψ', such that

$$\left.\begin{aligned} u' &= -\frac{\partial \psi'}{\partial z} \\ w' &= \frac{\partial \psi'}{\partial x} \end{aligned}\right] \tag{50}$$

Upon substituting from (50) in (48.1), the latter becomes

$$\left(\frac{\partial}{\partial t} + U'\frac{\partial}{\partial x}\right)\nabla^2\psi' = 0 \tag{51}$$

$$\text{where } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.$$

Let the solution of ψ' be of the form

$$\psi' = \varphi'(z)e^{i(mx-\sigma t)}$$

where $\varphi'(z)$ is a function of z only. Upon substituting in (51), the following differential equation for φ' is obtained

$$\frac{d^2\varphi'}{dz^2} - m^2\varphi' = 0.$$

This has the following solution

$$\varphi' = C_1'e^{mz} + C_2'e^{-mz}.$$

Because φ' has to remain finite when z becomes infinite, it follows that

$$C_1' = 0, \qquad C_2' = C'$$

hence

$$\varphi' = C'e^{-mz}$$

and therefore

$$\psi' = C'e^{-mz}e^{i(mx-\sigma t)} \tag{52}$$

and, from (50),

$$u' = c'me^{-mz}e^{i(mx-\sigma t)} \quad \text{(a)}$$

and

$$w' = iC'me^{-mz}e^{i(mx-\sigma t)}. \quad \text{(b)} \qquad (53)$$

Also, from (48), it is found that

$$P' = C'\rho'(\sigma - mU')e^{-mz}e^{i(mx-\sigma t)}. \quad \text{(c)}$$

The boundary conditions at the surface which separates the upper fluid (the air) from the lower fluid (the water) is

$$P + p - (P' + p') = 0 \tag{54}$$

where the capitals stand for the undisturbed conditions and the small letters for the perturbation quantities. Under the previous assumptions (54) may be written in the following differential form

$$\frac{\partial}{\partial t}(p - p') + \begin{bmatrix} U \\ U' \end{bmatrix} \frac{\partial}{\partial x}(p - p') + \begin{bmatrix} w \\ w' \end{bmatrix} \frac{\partial}{\partial z}(P - P') = 0, \tag{55}$$

$$\text{at} \quad z = 0.$$

Because both fluids are assumed to have no vertical accelerations in the undisturbed case, the undisturbed pressures are given by the hydrostatic equation, and therefore

$$\frac{\partial}{\partial z}(P - P') = -g(\rho - \rho'). \tag{56}$$

Upon substituting in (55) from (30), (53), and (56), first for the upper fluid then for the lower, the following two equations are obtained

$$(\sigma - mU')\alpha\rho[(c - V_0)S_{02} + V_0S_{01}]C + [(\sigma - mU')\rho' m(c - U') - g(\rho - \rho')m]C' = 0$$

$$\{(\sigma - mV_0)\alpha\rho[(c - V_0)S_{02} + V_0S_{01}] - mS_{01}g(\rho - \rho')\}C + (\sigma - mV_0)\rho' m(c - U')C' = 0$$

where $c = \dfrac{\sigma}{m}$ is the phase velocity of waves; and, upon equating the ratio of the constants, the following frequency equation is found

$$m\rho'(c - V')^2 + \alpha\rho(c - V_0)^2\frac{S_{02}}{S_{01}} + \alpha V_0\rho(c - V_0) - g(\rho - \rho') = 0. \tag{57}$$

Solving this equation for c, the following is found

$$c = \frac{\rho' U' + \dfrac{\alpha S_{02}}{mS_{01}}\rho V_0 - \dfrac{1}{2}\dfrac{\alpha}{m}\rho V_0}{\rho' + \dfrac{\alpha S_{02}}{mS_{01}}\rho}$$

$$\pm\sqrt{\frac{g(\rho - \rho')}{m\left(\rho' + \dfrac{\alpha S_{02}}{mS_{01}}\rho\right)} - \frac{m\alpha\dfrac{S_{02}}{S_{01}}\rho\rho'(U' - V_0)^2 + m\alpha\rho\rho' V_0(U' - V_0) - (\frac{1}{2}\alpha\rho V_0)^2}{\left(m\rho' + \alpha\dfrac{S_{02}}{S_{01}}\rho\right)^2}} \tag{58}$$

It may be remarked that, if in this equation ρ' is set equal to zero, the equation reduces to

$$c = V_0 - \left(\tfrac{1}{2}V_0\frac{S_{01}}{S_{02}}\right) \pm \sqrt{\frac{g}{\alpha}\frac{S_{01}}{S_{02}} + \left(\tfrac{1}{2}V_0\frac{S_{01}}{S_{02}}\right)^2}, \tag{32.1}$$

which is the same as EQUATION 32. Furthermore, if the lower fluid is assumed to have a constant basic current, *i.e.* if $\alpha \to 0$, then $\dfrac{S_{02}}{S_{01}}\alpha \to m$ and the equation reduces to

$$c = \frac{\rho' U' + \rho V_0}{\rho' + \rho} \pm \sqrt{\frac{g}{m}\frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho\rho'(U' - V_0)^2}{(\rho + \rho')^2}}, \tag{58.2}$$

which is the well known equation for the velocity of waves at the inner surface separating two incompressible fluids (see Lamb[6] p. 373).

It can be seen from EQUATION 58 that the stability of the waves depends upon the value of the radical. The waves will be stable if

$$\frac{g}{m}\,\frac{(\rho-\rho')\left(m\rho'+\alpha\frac{S_{02}}{S_{01}}\rho\right)}{\rho\rho'} \geqq \left[\alpha\frac{S_{02}}{S_{01}}(\Delta U)^2+\alpha V_0(\Delta U)-\left(\frac{1}{4}\frac{\alpha^2}{m}\frac{\rho}{\rho'}V_0^2\right)\right] \tag{59}$$

where

$$\Delta U = U' - V_0.$$

Or if

$$\lambda \geqq \frac{2\pi}{g}\,\frac{\frac{\alpha}{m}\frac{S_{02}}{S_{01}}\rho\rho'(\Delta U)^2+\frac{\alpha}{m}\rho\rho' V_0(\Delta U)-\left(\frac{1}{2}\frac{\alpha}{m}\rho V_0\right)^2}{(\rho-\rho')\left(\rho'+\frac{\alpha S_{02}}{mS_{01}}\rho\right)} \tag{60}$$

where λ is the wave length.

The inequality (60) is not readily susceptible to computations, because of the appearance of the unknown m in the right hand side, and because the ratio $\frac{S_{02}}{S_{01}}$ involves the same quantity in the form of a sum of terms in the ascending powers of m. A certain amount of discussion is, however, possible. EQUATION 60 may be compared with the corresponding equation which is derived from (58.2). There, it is found that the condition for stability is given by the inequality

$$\lambda_s \geq \frac{2\pi}{g}\,\frac{\rho\rho'(\Delta U)^2}{(\rho-\rho')(\rho+\rho')} \tag{61}$$

Since $\frac{\alpha}{m}\frac{S_{02}}{S_{01}}$ is not very different from unity, the denominator of both inequalities are nearly equal. The first term on the right hand side of (60) is therefore nearly equal to the right hand side of (61). Hence, for a first approximation the inequality (60) can be written in the following form

$$\lambda \geqq \lambda_s + \frac{\frac{\alpha}{m}\rho\rho' V_0(\Delta U)-\left(\frac{1}{2}\frac{\alpha}{m}\rho V_0\right)^2}{(\rho-\rho')(\rho+\rho')}. \tag{60.1}$$

(60.1) gives for the critical wave length a value which is somewhat higher than that computed on the basis of the simpler inequality (61). Thus, for $U' = 10$ m/sec, $\rho = 1$, $\rho = 1.3 \times 10^{-3}$ and $\alpha = 10^{-3}$ c.g.s. units, the critical wave length for $V = 15$ cm/sec is 8.124 cms, and for $V = 10^2$ cm/sec it is 9.683 cms, whereas (61) gives 8.1 cm and 6.25 cm respectively. The correction, therefore, becomes more pronounced as the surface current increases.

Summary

In this paper, a first attempt is made towards a solution of wave motion superposed on a current which is caused by winds. The current is assumed to follow the Ekman's spiral in speed but to remain constant in direction. Frequency equations are derived in the form of series solutions, and approximate computations on the basis of these series are made. The results are compared with gravitational waves that are travelling on a constant current. The stability of surface waves is also discussed. The assumption is made that the wind velocity is constant with height.

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