THE "MUSICAL" SOUND EMITTED BY A TORNADO

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ABSTRACT

This paper attempts to explain the "whining" or "hissing" sound reported from tornadoes. The leading hypothesis is that the air masses involved in the tornado circulation execute some free vibrations. It is found that these vibrations may be in the audible range for a small vortex whose radius is of the order of 10 m. or less. A formula is obtained which relates the frequency of the tone to the inner radius of the vortex. This formula is amenable to experimental verification.

1. DESCRIPTIVE INTRODUCTION

Several observers have stated that a tornado is usually attended by the emission of sound. This sound may be of the nature of "noise", or it may be of the nature of a "musical tone", the terms "noise" and "musical tone" being used in their physical sense. Thus a noise is a sound produced by some irregular vibrations that have no well defined overall frequency. A musical tone, on the other hand, is a sound associated with well defined regular vibrations which have a more or less well defined frequency. According to these definitions the rumbling sound of a train, that of thunder, that of a cannon, or the roar of a lion are noises. The sound of a tuning fork, or of a siren, or that of a flying bee may be classified as musical sounds, or tones.

Flora ([5], p. 3) states that

> Destruction starts when this cloud [i.e. the pendent cloud of the tornado] dips to the ground with a terrific roar, often described as resembling the noise of a thousand railway trains crossing trestles, or the sound of a cannon prolonged for a few minutes. Observers have also mentioned a peculiar whining sound like the buzzing of a million bees, which is usually heard when the cloud is high in the air. It is commonly drowned out by the roar when the cloud makes contact with the ground and destruction begins.

Flora (p. 11) also relates the description given by an observer of the noise produced by the Omaha, Nebr. tornado of March 23, 1913:

> The noise was like ten million bees, plus a roar that beggers description.

Another observer, describing the noise he heard from the Dodge City, Kans. tornado of June 22, 1928, is quoted by Flora (p. 13):

> At last the great shaggy end of the funnel hung directly over head. . . . There was a screaming, hissing sound coming directly from the end of the funnel. . . . Around the rim of the great vortex [about 50–100 ft. diameter] small tornadoes were constantly forming and breaking away. . . . It was these that made the hissing sound.

An observer describing the sound emitted by a tornado that formed in Germany on June 17, 1931 is quoted as follows ([5], p. 181):

> The funnel-shaped cloud was gray-black in color and advanced rapidly, accompanied by a noise "like the howl of dozens of sirens."

Brooks [4] describes the sound of a tornado as follows:

> A tornado reaching the ground produces a roaring or buzzing sound which has been heard as long as one hour before it arrived.

Then he makes the following noteworthy remark:

> As this noise still occurs when a whirl is aloft (though to a lesser extent), it is not due entirely to the destruction being caused by the wind, but is due also to vibrations created by frictional effects in the strong wind shear of the whirl. Such sounds are augmented by long rolls of thunder, which may overlap to make a nearly continuous background of rumble.

It thus appears, from the quotations cited above and from other descriptions, that the sound usually attending a tornado is either one or a mixture of two distinct phenomena: a noise normally described as a roar or a rumble, and a musical tone normally described as a buzz or a whine. The noise may be attributed to various sources, such as the sound associated with destruction or collision of flying debris, or even the thunder that accompanies the twister. If it be established by observation that the winds may reach sonic or supersonic speeds (Flora [5] p. 13), then the noise may also be attributed to the shock waves that must form in the supersonic regions of the circulation. The present writer [1] has already suggested that supersonic flow may be possible with a proposed tornado model. Anderson and Freier [3] have offered an explanation of the loud roar based on the possibility of the existence of circulating acoustic waves in a tornado vortex. Their explanation does not necessarily require a supersonic region. Loud noises may form as a result of the concentration of energy brought about by converging sound waves.

It is the object of the present article to study the second class of sound—the musical sound. A complete and thorough discussion of this phenonenon is difficult because of lack of careful observations, not to mention the difficulties encountered in an exact mathematical analysis. Accurate measurements of the various elements that describe a tornado are still beyond present facilities. Thus, for instance, we do not have direct measurements of the radius of the twister, the exact wind distribution, the pressure and temperature variations, or the vertical cross-section of the vortex. The sound phenomena themselves have been mentioned only in a general descriptive way. Observers even differ about the very existence of this phenomenon. Nevertheless, it is felt that a preliminary theoretical study is desirable for two main reasons. First, it may be possible, by theoretical reasoning, to establish the possibility of these sounds and to give a general description of their nature and the conditions under which they may or may not exist. The second is to inquire into the possibility of using these sounds when they exist as an additional tool of observation. It is always true that it is a relatively easy matter to obtain a rather quick and accurate estimate of the pitch of a musical tone. It is also true that, knowing the general nature of a sound source, one can tell quite a few things about that source from the pitch of the tone it emits. It may therefore be possible to tell some of the properties of the tornado just by making some measurements of the regular sound it creates. It is felt that the findings of such a theoretical treatment may serve to call the attention of observers to this tool which nature provides and someone may be able to record it and compare it with the results to be established.

While the present writer differs with Brooks about the mechanism responsible for the creation of the sound, he agrees with him on the basic assumption that this sound is caused by the vibrations of the air masses as a whole. Every system subject to equivalent restoring force and damping that is smaller than the critical value of aperiodicity, when displaced from a state of equilibrium, oscillates before attaining a new state of equilibrium or before returning to its old state. The air masses involved in the tornado circulation are no exception. Because these air masses are subject to various disturbing forces oscillation may be expected. It will be shown in the present article that, under some appropriate conditions, a tornado may execute some short-wave vibrations. The frequency of the normal vibrations may lie in the audible range, thus making a tornado act as a huge sound source. Because of the regularity of these vibrations they belong to the musical class, despite the fact that this kind of music may not be a welcome one.

2. WORKING MODEL

According to present standing theories, a widely accepted model of a tornado is that of a Rankine combined vortex superposed on a sink in a compressible atmosphere (Abdullah [1]). Such a model calls for radial as well as transversal flows. In an idealized circulation caused by such a combination the temperature may be expected to have a horizontal as well as a vertical gradient. Furthermore, the proper tornado circulation may vary with height, both in character and in size. The sink and the sense of vorticity may be expected to reverse themselves at greater heights.

Because of the mathematical difficulties encountered in dealing with such a complicated model, some simplifying assumptions are made in the present treatment. The working model to be dealt with here is that of a pure Rankine vortex imbedded in an isothermal compressible atmosphere. Radial and vertical velocities are therefore neglected in the initial conditions. It is thus assumed that a tornado consists of two distinct regions. The interior region is a right cylindrical column of air rotating around its geometrical axis as a solid body. The exterior region consists of all the rest of the atmosphere which is affected by the circulation. The flow in this region follows the hyperbolic law. The vortex may be assumed stationary relative to the ground. Friction and the rotation of the earth are neglected.

When this vortex is disturbed the air particles may move in all three dimensions of space. Waves may form and propagate in both the horizontal and vertical directions. Because interest is centered around musical tones, the oscillations will be assumed to be harmonic in time. The amplitude of these oscillations may be small, so that shock waves and related phenomena are excluded. The motion of the disturbed vortex is that of a freely vibrating system. No forcing mechanism is postulated.

3. MATHEMATICAL ANALYSIS

Let a cylindrical polar system of coordinates be chosen as shown in figure 1. The origin of the coordinates is at the geometrical center of the base of the vortex column, and OX is an arbitrarily chosen fixed horizontal direction. The z-axis is vertical and points upward. In the undisturbed case the motion is strictly horizontal and in the tangential direction, so that it may be described by the following relations:

$$U_i=\Omega r; \qquad 0\leq r\leq a$$
$$U_e=k/r; \qquad a\leq r\leq \infty \qquad (1)$$
$$W=0, \qquad V=0$$

where U is the tangential component of velocity, the indices i and e denote the interior and exterior regions, respectively, and W and V are the vertical and radial components of velocity. Capital letters refer to the undisturbed quantities. a is the radius of the interior region, and Ω and k are constants of proportionality.

If the assumption is made that there is no discontinuity in the velocity vector at the boundary between the two regions, the following relation between Ω and k may be deduced:

$$k=\Omega a^2 \tag{2}$$

In the working model it has been assumed that the atmosphere is isothermal. The additional assumption will be made here that all subsequent changes caused by the vibrations follow an isothermal law of expansion. In other words, the case under present consideration is that of an autobarotropic model. Although it is known that in acoustic vibrations the particles follow an adiabatic law of expansion, the isothermal assumption is adopted because it greatly facilitates the analysis while it does not affect the results appreciably, as will be shown later by induction. With this assumption, the Newtonian speed of sound, c, may be defined by the following equation:

$$c^2=\frac{dP}{d\rho}=RT \tag{3}$$

where the letters have their usual meanings, and T is a constant.

The height of the homogeneous atmosphere is defined by the following relation (see, for example, Haurwitz [6]):

$$H=\frac{RT}{g}=\frac{c^2}{g}=\text{constant} \tag{4}$$

With the usual perturbation assumptions, and with the undisturbed quantities denoted by capital letters and the perturbation quantities by small letters, the equations of motion for the undisturbed state, in polar coordinates, are found to be

$$\left.\begin{aligned}\frac{U^2(r)}{r}&=\frac{1}{\rho_0}\frac{\partial P}{\partial r},\\ 0&=-\frac{1}{\rho_0}\frac{\partial P}{\partial z}-g\end{aligned}\right\} \tag{5}$$

where ρ_0 is the undisturbed density. The equation of continuity is satisfied identically for this case.

The perturbation equations of motion and continuity for infinitesimal vibrations are the following:

$$\left.\begin{aligned}&\left(\frac{\partial}{\partial t}+\frac{U}{r}\frac{\partial}{\partial \theta}\right)u+\left(\frac{dU}{dr}+\frac{U}{r}\right)v=-c^2\frac{\partial\hat{p}}{r\partial\theta} && \text{(a)}\\ &\left(\frac{\partial}{\partial t}+\frac{U}{r}\frac{\partial}{\partial \theta}\right)v-\frac{2Uu}{r}=-c^2\frac{\partial\hat{p}}{\partial r} && \text{(b)}\\ &\left(\frac{\partial}{\partial t}+\frac{U}{r}\frac{\partial}{\partial \theta}\right)w=-c^2\frac{\partial\hat{p}}{\partial z} && \text{(c)}\\ &\left(\frac{\partial}{\partial t}+\frac{U}{r}\frac{\partial}{\partial \theta}\right)\hat{p}-\epsilon w+\left(\frac{U^2}{c^2}+1\right)\frac{v}{r}+\frac{\partial w}{\partial z}+\frac{\partial v}{\partial r}+\frac{\partial u}{r\partial\theta}=0 && \text{(d)}\\ &\text{where}\quad \hat{p}=\frac{p}{P} && \text{(e)}\\ &\text{and}\quad \epsilon=\frac{1}{H} && \text{(f)}\end{aligned}\right\} \tag{6}$$

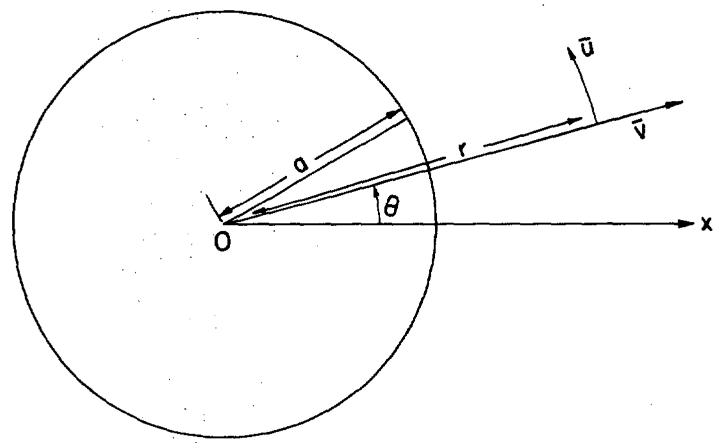

FIGURE 1.—A schematic representation of the horizontal cross-section of the vortex under consideration. OX is an arbitrarily fixed line from which the angular distance θ is measured. a is the radius of the interior region.

To eliminate t and θ the following values will be assumed for the unknown variables:

$$\left.\begin{aligned}u&=u(r,z)\cos(\alpha t-\beta\theta)\\ v&=v(r,z)\sin(\alpha t-\beta\theta)\\ w&=w(r,z)\sin(\alpha t-\beta\theta)\\ \text{and}\quad \hat{p}&=\hat{p}(r,z)\cos(\alpha t-\beta\theta)\end{aligned}\right\} \tag{7}$$

Upon insertion of these values in (6) the following equations result:

$$\left.\begin{aligned}&-\left(\alpha-\frac{U\beta}{r}\right)u+\left(\frac{dU}{dr}+\frac{U}{r}\right)v=-\frac{c^2\beta}{r}\hat{p} && \text{(a)}\\ &\left(\alpha-\frac{U\beta}{r}\right)v-\frac{2Uu}{r}=-c^2\frac{\partial\hat{p}}{\partial r} && \text{(b)}\\ &\left(\alpha-\frac{U\beta}{r}\right)w=-c^2\frac{\partial\hat{p}}{\partial z} && \text{(c)}\\ &-\left(\alpha-\frac{U\beta}{r}\right)\hat{p}-\epsilon w+\left(\frac{U^2}{c^2}+1\right)\frac{v}{r}+\frac{\partial w}{\partial z}+\frac{\partial v}{\partial r}+\frac{\beta u}{r}=0 && \text{(d)}\end{aligned}\right\} \tag{8}$$

It may be remarked that the functional dependence of the unknown variables upon r and z has not been written down explicitly, since no confusion is expected to arise.

From (a) and (b) of (8) the following values are obtained for u and v:

$$\left.\begin{aligned}&u=\frac{c^2}{D}\left[\left(\frac{dU}{dr}+\frac{U}{r}\right)\frac{\partial\hat{p}}{\partial r}-\frac{\beta}{r}\left(\alpha-\frac{U\beta}{r}\right)\hat{p}\right] && \text{(a)}\\ &v=\frac{c^2}{D}\left[\left(\alpha-\frac{U\beta}{r}\right)\frac{\partial p}{\partial r}-\frac{2\beta U}{r^2}\hat{p}\right] && \text{(b)}\\ &\text{where}\\ &D=\frac{2U}{r}\left(\frac{dU}{dr}+\frac{U}{r}\right)-\left(\alpha-\frac{U\beta}{r}\right)^2 && \text{(c)}\end{aligned}\right\} \tag{9}$$

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and from (c) of (8) the following value of w is found:

$$w=-\frac{c^2}{\left(\alpha-\frac{U\beta}{r}\right)}\frac{\partial\hat{p}}{\partial z} \tag{10}$$

Upon substitution from (9) and (10) in (8d) the following equation is found for $\hat{p}$:

$$L(\hat{p})\equiv-\frac{\partial^2\hat{p}}{\partial z^2}+A(r)\frac{\partial^2\hat{p}}{\partial r^2}+\epsilon\frac{\partial\hat{p}}{\partial z}+B(r)\frac{\partial\hat{p}}{\partial z}+E(r)\hat{p}=0 \tag{11}$$

where

$$\left.\begin{aligned}
A(r)&=\frac{1}{D}\left(\alpha-\frac{\beta U}{r}\right)^2 && \text{(a)}\\
B(r)&=\left(\frac{U^2}{c^2}+1\right)\frac{1}{rD}\left(\alpha-\frac{\beta U}{r}\right)^2-\frac{1}{D}\left[\frac{1}{D}\frac{dD}{dr}\left(\alpha-\frac{\beta U}{r}\right)^2\right.\\
&\quad\left.+\beta\left(\alpha-\frac{\beta U}{r}\right)\frac{d}{dr}\left(\frac{U}{r}\right)+2\beta\left(\alpha-\frac{\beta U}{r}\right)\frac{U}{r^2}\right]\\
&\quad+\frac{\beta}{rD}\left(\alpha-\frac{\beta U}{r}\right)\left(\frac{dU}{dr}+\frac{U}{r}\right) && \text{(b)}\\
E(r)&=-\frac{1}{c^2}\left(\alpha-\frac{\beta U}{r}\right)^2-\left(\frac{U^2}{c^2}+1\right)\left(\alpha-\frac{\beta U}{r}\right)\frac{2\beta}{rD}\left(\frac{U}{r^2}\right)\\
&\quad+\frac{2\beta}{D}\left(\alpha-\frac{\beta U}{r}\right)\left[\frac{1}{D}\frac{dD}{dr}\left(\frac{U}{r^2}\right)-\frac{d}{dr}\left(\frac{U}{r^2}\right)\right]\\
&\quad-\frac{\beta^2}{r^2D}\left(\alpha-\frac{\beta U}{r}\right)^2 && \text{(c)}
\end{aligned}\right\} \tag{12}$$

Let the following value be assumed for $\hat{p}$:

$$\hat{p}(r,z)=Z(z)R(r) \tag{13}$$

Substituting in (11), with the usual procedure of separation, results in the following two equations:

$$\frac{d^2Z}{dz^2}-\epsilon\frac{dZ}{dz}+\lambda^2Z=0 \tag{14}$$

$$A(r)\frac{d^2R}{dr^2}+B(r)\frac{dR}{dr}+(E(r)+\lambda^2)R=0 \tag{15}$$

where λ is the separation constant.

Equations (14) and (15) are the two basic equations to be solved for the two distinct regions.

4. SOLUTION OF THE EQUATION FOR THE VERTICAL COMPONENT

Equation (14) has the following solution:

$$Z=e^{\frac{\epsilon}{2}z}\left[C^1\cos\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z+C\sin\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z\right] \tag{16}$$

where C^1 and C are the two constants of integration.

The boundary condition to be satisfied at the horizontal ground is

$$w=0,\qquad \text{at } z=0$$

From (10) and (13) this condition becomes

$$\frac{dZ}{dz}=0,\qquad \text{at } z=0 \tag{17}$$

This condition serves to determine C^1 in terms of C, as a result of which equation (16) takes the following form:

$$Z=Ce^{\frac{\epsilon}{2}z}\left[\sin\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z-\left(\frac{4\lambda^2}{\epsilon^2}-1\right)^{1/2}\cos\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z\right] \tag{18}$$

If the additional plausible restriction is imposed that w may vanish at some other height, h, the admissible values of λ are found to be

$$\lambda_s=\left(\frac{s^2\pi^2}{h^2}+\frac{1}{(2H)^2}\right)^{1/2},\qquad s=0,1,2,\ldots \tag{19}$$

If h is identified with H, the height of the homogeneous atmosphere, this reaction takes the following form:

$$\lambda_s=\frac{1}{H}\left(s^2\pi^2+\frac{1}{4}\right)^{1/2},\qquad s=0,1,2,\ldots \tag{20}$$

The lowest value of λ_s is found to be $\lambda_0=1/2H$, and the higher overtones may be found by giving s the proper values.

5. SOLUTION OF THE EQUATION FOR THE HORIZONTAL COMPONENT

Because $U(r)$ appears in the coefficients of equation (15) its solution differs between the interior and the exterior regions. However, before the attempt is made to obtain these solutions an approximation will be made which greatly facilitates the analytical procedure.

The maximum value that the ratio U/c may attain is its value at the boundary between the two regions. Indirect observations indicate that in most cases this quantity is of the order of 1/2 or less. The quantity $(U/c)^2$ may therefore be neglected in comparison with unity without introducing serious error. It is felt that the simplifications introduced by making this approximation may justify its adoption despite the limitations it imposes upon the model.

6. SOLUTION APPROXIMATE TO THE INTERIOR REGION, $0\leq r\leq a$

In the interior region relation (1) gives for U the value Ωr. Inserting this in (12) and (9c), neglecting $(U/c)^2$ in comparison with unity, and substituting in (15), yield the following equation:

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(m^2-\frac{\beta^2}{r^2}\right)R=0 \tag{21}$$

where

$$m^2=\left[\lambda^2-\frac{(\alpha-\Omega\beta)^2}{c^2}\right]\left[\frac{4\Omega^2-(\alpha-\Omega\beta)^2}{(\alpha-\Omega\beta^2)}\right] \tag{22}$$

Equation (21) is the typical Bessel equation. The nature of the cylindrical functions that satisfy this equation depends upon the nature of m, whether it is real or imaginary. In anticipation of the discussion to be given later, m will be taken to be a real number. Equation (21) has, therefore, the following solution (see, for example, Abramowitz and Stegun [2]):

$$R=GJ_\beta(mr)+G'Y_\beta(mr), \qquad 0\leq r\leq a \tag{23}$$

where G and G' are the two constants of integration, and $J_\beta(mr)$ and $Y_\beta(mr)$ are, respectively, the Bessel functions of first and second kind.

Because $Y_\beta(mr)$ goes to infinity at $r=0$, and since the motion must be bounded at this point, G' must be zero. The appropriate solution is, therefore, the following:

$$R=GJ_\beta(mr), \qquad 0\leq r\leq a \tag{24}$$

Upon combining this value with (18), inserting in (13), then making use of (9), (10) and (7), we find the solutions relevant to the interior region to be the following, the constant G being absorbed in C:

$$\left.\begin{aligned}
u_i&=\frac{c^2Ce^{\frac{\epsilon}{2}z}}{[4\Omega^2-(\alpha-\Omega\beta)^2]}\left[2\Omega\frac{d}{dr}J_\beta(mr)-\frac{\beta}{r}(\alpha-\Omega\beta)J_\beta(mr)\right]\\
&\quad\times\left[\sin\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z-\left(\frac{4\lambda^2}{\epsilon^2}-1\right)^{1/2}\right.\\
&\quad\left.\cos\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z\right]\cos(\alpha t-\beta\theta) \qquad \text{(a)}\\
v_i&=\frac{c^2Ce^{\frac{\epsilon}{2}z}}{[4\Omega^2-(\alpha-\Omega\beta)^2]}\left[(\alpha-\Omega\beta)\frac{d}{dr}J_\beta(mr)\right.\\
&\quad\left.-\frac{2\Omega\beta}{r}J_\beta(mr)\right]\times\left[\sin\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z-\left(\frac{4\lambda^2}{\epsilon^2}-1\right)^{1/2}\right.\\
&\quad\left.\cos\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z\right]\sin(\alpha t-\beta\theta) \qquad \text{(b)}\\
w_i&=-\frac{2c^2\lambda^2Ce^{\frac{\epsilon}{2}z}}{\epsilon(\alpha-\Omega\beta)}J_\beta(mr)\sin\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z\cdot\sin(\alpha t-\beta\theta) \qquad \text{(c)}\\
\hat{p}_i&=\frac{p_i}{P_i}=Ce^{\frac{\epsilon}{2}z}J_\beta(mr)\left[\sin\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z\right.\\
&\quad\left.-\left(\frac{4\lambda^2}{\epsilon^2}-1\right)^{1/2}\cos\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z\right]\cos(\alpha t-\beta\theta) \qquad \text{(d)}
\end{aligned}\right\} \tag{25}$$

7. SOLUTION APPROPRIATE TO THE EXTERIOR REGION, $a\leq r\leq\infty$

In the exterior region relations (1) give for U the value $U_e=k/r$. When this value is inserted in the relevant equations, (15) takes the following form:

$$\frac{d^2R}{dr^2}+\left[\frac{1}{r}-\frac{4\beta k}{r^3\left(\alpha-\frac{\beta k}{r^2}\right)}\right]\frac{dR}{dr}+\left[\frac{\left(\alpha-\frac{\beta k}{r^2}\right)^2}{c^2}-\lambda^2-\frac{\beta^2}{r^2}+\frac{4\beta k}{r^4\left(\alpha-\frac{\beta k}{r^2}\right)}+\frac{8\beta^2k^2}{r^6\left(\alpha-\frac{\beta k}{r^2}\right)^2}\right]R=0 \tag{26}$$

This equation may be simplified by considering the orders of magnitude of the various quantities and neglecting terms of smaller magnitudes.

Thus it may be noticed that powers of r appear in the denominators of some terms which will be shown to be much smaller than the rest. The minimum value that r can attain in the exterior region is its value at the boundary between the two regions. A representative magnitude of this quantity is 10^4 cm. However, it will be shown that even for the still smaller value of 10^3 cm. the approximations to be made are justifiable.

Take the maximum value of U to be of the order of 10^4 cm. sec.$^{-1}$ It follows that k is of the order of 10^7 c.g.s. units. β may be of the order of 10. Because we are looking for sound waves α may be of the order of 10^3 sec.$^{-1}$ This makes the frequency of the order 10^2 sec.$^{-1}$

The two terms in the coefficient of dR/dr have the orders of magnitude

$$r^{-1}\to O(10^{-3}) \text{ and } \frac{4\beta k}{r^3\left(\alpha-\frac{\beta k}{r^2}\right)}\to O(10^{-4}) \text{ or less}$$

Hence the second term may be neglected. Similarly, if in the coefficient of R quantities of order less than 10^{-5} are neglected, equation (26) may be put in the following approximate form

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(\delta^2-\frac{\mu^2}{r^2}\right)R=0 \tag{27}$$

where

$$\delta^2=\frac{\alpha^2}{c^2}-\lambda^2$$

and

$$\mu^2=\beta^2+\frac{2\beta k\alpha}{c^2} \tag{28}$$

Equation (27) is again in the typical form of Bessel equation. The quantity μ is always a real number. δ is also real for small values of λ, which is usually the case, as may easily be inferred. Hence the solution relevant to these conditions is the following:

$$R=A_1J_\mu(\delta r)+A_2Y_\mu(\delta r) \tag{29}$$

where A_1 and A_2 are the two constants of integration.

Upon making use of (29), following the same procedure as in the previous case, and letting the constant C be absorbed in A_1 and A_2, we find the following solutions for the various variables:

$$\left.\begin{aligned}
u_e&=\frac{c^2\beta e^{\frac{\epsilon}{2}z}}{r\left(\alpha-\frac{\beta k}{r^2}\right)}[A_1J_\mu(\delta r)+A_2Y_\mu(\delta r)]\left[\sin\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z\right.\\
&\qquad\left.-\left(\frac{4\lambda^2}{\epsilon^2}-1\right)^{1/2}\cos\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z\right]\cos(\alpha t-\beta\theta)\quad\text{(a)}\\
v_e&=-\frac{c^2e^{\frac{\epsilon}{2}z}}{\left(\alpha-\frac{\beta k}{r^2}\right)^2}\left[\left(\alpha-\frac{\beta k}{r^2}\right)[A_1\frac{d}{dr}J_\mu(\delta r)+A_2\frac{d}{dr}Y_\mu(\delta r)]\right.\\
&\qquad\left.-\frac{2\beta k}{r^3}[A_1J_\mu(\delta r)+A_2Y_\mu(\delta r)]\right]\times\left[\sin\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z\right.\\
&\qquad\left.-\left(\frac{4\lambda^2}{\epsilon^2}-1\right)^{1/2}\cos\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z\right]\sin(\alpha t-\beta\theta)\quad\text{(b)}\\
w_e&=-\frac{2c^2\lambda^2e^{\frac{\epsilon}{2}z}}{\epsilon\left(\alpha-\frac{\beta k}{r^2}\right)}[A_1J_\mu(\delta r)+A_2Y_\mu(\delta r)]\\
&\qquad\sin\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z\sin(\alpha t-\beta\theta)\quad\text{(c)}\\
\hat{p}_e&=\frac{p_e}{P_e}=e^{\frac{\epsilon}{2}z}[A_1J_\mu(\delta r)+A_2Y_\mu(\delta r)]\left[\sin\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z\right.\\
&\qquad\left.-\left(\frac{4\lambda^2}{\epsilon^2}-1\right)^{1/2}\cos\left(\lambda^2-\frac{\epsilon^2}{4}\right)^{1/2}z\right]\times\cos(\alpha t-\beta\theta)\quad\text{(d)}
\end{aligned}\right\}\quad(30)$$

8. BOUNDARY CONDITIONS AND FREQUENCY EQUATION

It remains to fit the boundary conditions and obtain a frequency equation for these vibrations. In order to write down a simplified version for the boundary conditions pertinent to the motion under consideration, it will be assumed that no slipping be permissible at the boundary surface between the two regions. The same condition has been used by Kelvin [7] in discussing the incompressible case of the present model.

The boundary conditions may therefore be written as follows:

$$\left.\begin{aligned} v_i&=v_e &\text{(a)}\\ u_i&=u_e &\text{(b)}\\ w_i&=w_e &\text{(c)}\end{aligned}\right\}\text{ at } r=a \qquad (31)$$

The first condition also follows from continuity considerations. From the third of these conditions and (10) it follows that

$$\hat{p}_i=\hat{p}_e;\ r=a \qquad (31d)$$

From these conditions and equations (25) and (30) the following relations are found among the constants A_1, A_2, and C:

$$\frac{A_1}{A_2}=-\frac{Y_\mu(\delta a)+\dfrac{\delta(\alpha-\Omega\beta)[4\Omega^2-(\alpha-\Omega\beta)^2]Y'_\mu(\delta a)J_\beta(ma)}{m(\alpha-\Omega\beta)^3J'_\beta(ma)-\dfrac{8\beta\Omega^3}{a}J_\beta(ma)}}{J_\mu(\delta a)+\dfrac{\delta(\alpha-\Omega\beta)[4\Omega^2-(\alpha-\Omega\beta)^2]J'_\mu(\delta a)J_\beta(ma)}{m(\alpha-\Omega\beta)J_\beta(ma)-\dfrac{8\beta\Omega^3}{a}J_\beta(ma)}} \qquad (32a)$$

and

$$\frac{C}{A_2}=-\frac{Y_\mu(\delta a)+\dfrac{\delta(\alpha-\Omega\beta)[4\Omega^2-(\alpha-\Omega\beta)^2]Y'_\mu(\delta a)J_\beta(ma)}{m(\alpha-\Omega\beta)[4\Omega^2-(\alpha-\Omega\beta)^2]J'_\mu(\delta a)J_\beta(ma)}}{J_\mu(\delta a)+\dfrac{\delta(\alpha-\Omega\beta)[4\Omega^2-(\alpha-\Omega\beta)^2]J'_\mu(\delta a)J_\beta(ma)}{m(\alpha-\Omega\beta)^3J'_\beta(ma)-\dfrac{8\beta\Omega^3}{a}J_\beta(ma)}}$$

$$\frac{J_\mu(\delta a)}{J_\beta(ma)}+\frac{Y_\mu(\delta a)}{J_\beta(ma)} \qquad (32b)$$

where

$$\left.\begin{aligned} J'_\mu(\delta a)&=\frac{d}{d(\delta r)}J_\mu(\delta r)\\ \text{and}\quad Y'_\mu(\delta a)&=\frac{d}{d(\delta r)}Y_\mu(\delta r)\end{aligned}\right\}\text{ at } r=a$$

Relations (32) make it possible to eliminate two of the integration constants, leaving only one arbitrary constant which is a measure of the amplitude of vibrations.

From the foregoing relations and equations (25a) and (30a), condition (31b) gives the following equation:

$$\frac{\beta}{a}[\alpha-(\beta+3)\Omega]J_\beta(ma)=m(\alpha-\Omega\beta)J_{\beta+1}(ma) \qquad (33)$$

This is the frequency equation for the vibrations under study.

9. SPECIAL CASE OF RADIALLY SYMMETRICAL VIBRATIONS

The solutions presented in the foregoing analysis describe the general motion subject to the postulated assumptions. However, the main object of the present article is to study the capability of a tornado to execute high-frequency free vibrations, and to discuss the conditions under which these vibrations lie in the audible range. It is therefore appropriate to limit the discussions to the lowest possible frequencies.

The lowest order of vibrations in the tangential direction is that described by setting $\beta=0$. This makes the motion radially symmetrical and independent of θ. The particles composing the same cylindrical ring vibrate in phase with each other.

The solutions for this specially important limiting case may readily be obtained from the relevant equations by giving β the value 0. The frequency equation (33) simplifies to the following:

$$J_1(ma)=0 \qquad (34)$$

From a consideration of the orders of magnitude it may easily be seen that ma is a large number. Equation (34) may therefore be written in the following asymptotic form:

$$J_1(ma) \xrightarrow[ma\to\infty]{} \left(\frac{2}{\pi ma}\right)^{1/2} \cos\left(ma-\frac{3\pi}{4}\right)=0 \qquad (35)$$

Hence

$$m=\frac{\pi}{4a}(4j+5); \qquad j=0, 1, 2 \ldots \qquad (36)$$

From (22), with $\beta=0$, the value of m is found to be

$$m=\left[\left(\frac{\alpha^2}{c^2}-\lambda^2\right)\left(\frac{\alpha^2-4\Omega^2}{\alpha^2}\right)\right]^{1/2} \qquad (37)$$

Upon combining this with (36) and solving for α, then noting that the pitch ν has the value $\alpha/2\pi$, we obtain the following value for $\nu_{0,j}$

$$\nu_{0,j}=\frac{c\sqrt{2}}{4\pi}\left\{\left[\lambda^2+\frac{4\Omega^2}{c^2}+\frac{\pi^2}{16a^2}(4j+5)^2\right] + \left[\left[\lambda^2+\frac{4\Omega^2}{c^2}+\frac{\pi^2}{16a^2}(4j+5)^2\right]^2-\frac{16\Omega^2\lambda^2}{c^2}\right]^{1/2}\right\}^{1/2} \qquad (38)$$

where the subscripts 0, j are written down to indicate that the frequency is that corresponding to $\beta=0$ and the chosen value of j.

In order to obtain an idea about the audibility of these vibrations it may be mentioned that experiments have shown that the normal human ear can detect a sound whose frequency is as low as 20 sec.$^{-1}$ (see, for example, Pollack [8]). To evaluate the expression given in (38) let us start by assuming that the temperature of the isothermal atmosphere is 273° K. The values of H and c corresponding to this temperature are, approximately, 8.10^5 cm. and 2.8×10^4 cm. sec.$^{-1}$, respectively. If the height, h, appearing in equation (19) be identified with the height of the homogeneous atmosphere, H, the value of λ corresponding to $s=1$ is 0.394×10^{-5} cm.$^{-1}$

Let the maximum undisturbed velocity be 10^4 cm. sec.$^{-1}$ The quantity Ω then has the value 10^4a^{-1}. When these values are substituted in equation (38) it is immediately seen that λ_1^2 and $16\Omega^2\lambda_1^2/c^2$ are at least two orders of magnitude smaller than the rest of the terms. If these quantities are neglected, equation (38) reduces to the following:

$$\nu_{0,j}=\frac{c}{2\pi a}\left[\frac{4U^2}{c^2}+\frac{(4j+5)^2\pi^2}{16}\right]^{1/2} \qquad (39)$$

An immediate result of this equation is that the pitch is inversely proportional to the radius of the interior region, a result which agrees with physical speculation.

The frequency of the fundamental mode of vibrations, $\nu_{0,f}$, may be obtained from (39) by giving j the value 0. Hence

$$\nu_{0,f}=\frac{c}{2\pi a}\cdot\left(\frac{4U^2}{c^2}+\frac{25\pi^2}{16}\right)^{1/2} \qquad (40)$$

It follows from this equation that the largest value of a that may give rise to audible fundamental tone is

$$a_{\max,f}=\frac{c}{2\pi\nu_{0,f}}\left(\frac{4U^2}{c^2}+\frac{25\pi^2}{16}\right)^{1/2} \qquad (41)$$

Upon making $\nu_{0,f}=20$, and substituting for U and c their assumed values, we find that $a_{\max,f}=9.25$ m. It would seem therefore that the fundamental tone is only audible in small vortices whose interior region is of an order of magnitude less than 10 m. It is known that the fundamental contains the maximum energy, and hence it is the mode which determines the overruling pitch. However, if the frequency of the fundamental lies in the silent domain, higher overtones may still be audible. Thus the first overtone corresponding to $j=1$ is represented by the following equation:

$$a_{\max,1}=\frac{c}{2\pi\nu_{0,1}}\left(\frac{4U^2}{c^2}+\frac{81\pi^2}{16}\right)^{1/2} \qquad (42)$$

Substituting the assumed values yields $a_{\max,1}=15.8$ m. Thus vortices with larger radii may be heard.

Equation (39) could be simplified further by neglecting the first term under the radical since it is normally smaller than the second. Equation (39) then takes the following approximate form:

$$\nu_{0,j}=\frac{(4j+5)c}{8a} \qquad (43)$$

The Newtonian speed of sound, c, appeared in this formula because, for the sake of simplicity, an isothermal law of expansion has been assumed. It is more correct, however, to assume an adiabatic law, in which case it may be expected to come out with the Laplacian speed which is more in agreement with measurements. If this speed is taken as 331 m. sec.$^{-1}$, (43) becomes

$$\nu_{0,j}=\frac{41.5(4j+5)}{a} \qquad (44)$$

where a is measured in meters.

The fundamental frequency is given by

$$\nu_f=\frac{207}{a} \qquad (45)$$

which is a very simple formula that may be used to determine one of the two unknowns, ν or a, when the other is known.

As an example, consider the case of the Dodge City tornado of June 22, 1928, which was mentioned in the introduction. The observer has estimated the diameter of the great vortex to be between 50 and 100 ft. Because of the obscurities to visibility that usually surround the interior region, it seems reasonable to take the lower value as the more likely one. This makes the radius about 7.5 m. The frequency of the fundamental tone corresponding to this radius is, from (45), 28 $sec.^{-1}$ which is in the audible range. The frequency of the first overtone is, from (44), 50 $sec.^{-1}$ which is well in the audible range.

10. CONCLUSIONS AND FURTHER REMARKS

(1) On the basis of the analysis cited above it may be concluded that tornadoes are capable of executing free vibrations with high frequencies. These frequencies may fall in the audible range if the radius of the vortex is small enough to satisfy the required conditions. This result may explain the whining or buzzing sound described by some observers. The lower harmonics of vibrations executed by larger vortices lie in the inaudible range. Hence only noise is likely to be heard from large vortices. Observers who have reported hearing the musical tone have also described the twisters from which the sound originated as being small in diameter. This agrees qualitatively with the findings of the present article.

(2) As a further evidence which is in agreement with the present findings it may be mentioned that the musical sound has mostly been reported from tornadoes that did not touch the ground. It is obviously true that any musical sound that may be emitted by tornadoes touching the ground may be drowned out by the louder noise associated with it. But it may also be remarked that tornadoes that do not touch the ground usually have small radii.

(3) The main result obtained by the theoretical analysis is that expressed by equation (39) which could be put in the simplified approximate form given in (44). This formula relates the frequency of the vibrations to the radius of the tornado. Because of the violence of the storm it is rather beyond present observational possibilities to measure the inner radius. However, it is much easier to measure the frequency. This could best be done by recording the vibrations and performing a harmonic analysis after filtering out the noise. This method may be applicable even if the frequencies lie in the infrasonic region. When the frequency is known, the inner radius may be computed. This is a tool which nature provides and which has not, so far, been used.

(4) Finally, the following remark may be made about the mathematical analysis. In the analysis the quantities m and μ were taken to be real numbers. The necessary condition for this to be true is approximately the following:

$$\lambda < \frac{\alpha}{c}$$

This condition leads to the following inequality:

$$L_z > L_r$$

where L_z is a wavelength in the vertical direction and L_r in the horizontal direction. This condition may be expected to hold in an atmosphere whose density decreases exponentially with height, while remaining constant in the horizontal direction. However, the possibility that this condition may be violated, and shorter vertical wavelengths may exist, cannot be ruled out completely. In that case the basic equations admit of solutions in terms of the modified Bessel functions of purely imaginary arguments, $I_m(mr)$ and $K_\mu(\mu r)$. A discussion of such solutions has not been attempted here because of the arbitrariness of the then resulting parameters, and because it was felt that the present assumptions may be more realistic.

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